

# Half-supersymmetric solutions in five-dimensional supergravity

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**ABSTRACT:** We present a systematic classification of half-supersymmetric solutions of gauged  $N = 2$ ,  $D = 5$  supergravity coupled to an arbitrary number of abelian vector multiplets for which at least one of the Killing spinors generate a time-like Killing vector.

**KEYWORDS:** Black Holes in String Theory, Supergravity Models.

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## 1. Introduction

There has been a considerable research effort in recent years devoted to the analysis and study of solutions of supergravity theories in various dimensions and in particular those obtained as low energy limits of superstring and M-theory. This effort is motivated by the important role that black holes and domain walls have played in some of the recent developments that took place in superstring theory. These include the conjectured equivalence between string theory on anti-de Sitter (AdS) spaces and certain superconformal gauge theories living on the boundary [1] known as the AdS/CFT correspondence, the

understanding of the microscopic analysis of black hole entropy [2] and the understanding of various duality symmetries relating string theories to each other and to M-theory. An interesting possibility that arises from the conjectured AdS/CFT correspondence is the ability to obtain information of the nonperturbative structure of field theories by studying dual classical gravitational configurations. A notable example in this direction is the Hawking-Page phase transition [3] which was interpreted in [4] as a thermal phase transition from a confining to a deconfining phase in the dual  $D = 4$ ,  $N = 4$  super Yang-Mills theory. Various interesting results using Anti-de Sitter black holes and their CFT duals have been obtained in recent years (see for example [5–10]).

In this paper we will focus on the study of supersymmetric solutions in five-dimensional  $N = 2$  gauged supergravity coupled to abelian vector multiplets [11]. These solutions are relevant for the holographic descriptions of four dimensional field theories with less than maximal supersymmetry. Explicit supersymmetric black holes for these theories were constructed in [12]. However, these solutions have naked singularities or naked closed time-like curves. Domain walls and magnetic strings were also constructed in [13]. Obviously one would like to study the general structure of supersymmetric solutions in five dimensions rather than some specific solutions based on a certain ansatz. The main purpose of this paper is to construct a systematic classification of half-supersymmetric solutions.

The first systematic classification of supersymmetric solutions, following the results of [15], was performed in [16] for minimal  $N = 2$  supergravity in  $D = 4$ . In [16] it was shown that supersymmetric solutions fall into two classes which depend on whether the Killing vector obtained from the Killing spinor is time-like or null. For the time-like case, one obtains the Israel-Wilson-Perjes class of solutions and the null solutions are pp-waves. Further generalizations were presented in [17]. More recently and motivated by the results of Tod, purely bosonic supersymmetric solutions of minimal  $N = 2$ ,  $D = 5$  were classified in [18]. The basic idea in this analysis is to assume the existence of a Killing spinor, (i.e., to assume that the solution preserves at least one supersymmetry) and construct differential forms as bilinears in the Killing spinor. Then Fierz identities and the vanishing of the supersymmetry transformation of the fermionic fields in a bosonic background provide a set of algebraic and differential equations for the spinor bilinear differential forms which can be used to deduce the form of the metric and gauge fields. Such a general framework provides a powerful method for obtaining many new solutions, in contrast to the earlier methods that start with an ansatz for the metric and assume certain symmetries for the solution from the outset. The strategy of [18] was used later to perform similar classifications of supersymmetric solutions in various supergravity theories. In particular, in [19] the classification of 1/4 supersymmetric solutions of the minimal gauged  $N = 2$ ,  $D = 5$  supergravity was performed.

Explicit supersymmetric asymptotically anti-de Sitter black hole solutions with no closed time loops or naked singularities were constructed for the minimal supergravity theory in [20]. The results of [19] for the time-like solutions were generalized in [21] to the non-minimal case where the scalar fields live on symmetric spaces and explicit solutions for the  $U(1)^3$  theory (with three  $R$ -charges) were also constructed. The constraint of symmetric spaces was relaxed in [22], where solutions with a null Killing vector in both

gauged and ungauged theories were also obtained.

In this paper we focus on the classification of half supersymmetric solutions in gauged  $N = 2, D = 5$  supergravity with vector multiplets. Half supersymmetric solutions have two Killing spinors from which one can construct two Killing vectors as bilinears in the Killing spinors. These vectors could be either time-like or null. Therefore one has to consider three cases depending on the nature of the Killing spinors and vectors considered. In our present work we will focus on the cases where the solutions contain at least one Killing spinor with an associated time-like Killing vector. In order to investigate supersymmetric solutions with more than one Killing spinor, it is very useful to express the Killing spinors in terms of differential forms [23–25]. Such a method, known now by the spinorial geometry method, has been very efficient in classifying solutions of supergravity theories in ten and eleven dimensions [26–29]. The spinorial geometry method has also been recently used to classify half-supersymmetric solutions in  $N = 2, D = 4$  supergravity [30].

We organize our work as follows. In section two, we present the basic structure of the theory of  $N = 2, D = 5$  gauged supergravity coupled to abelian vector multiplets and the equations of motion. In section three we express spinors in five dimensions as differential forms on  $\Lambda^*(\mathbb{R}^2) \otimes \mathbb{C}$ . We start with the generic form of the spinor and then use the gauge symmetries ( $U(1)$  and  $Spin(4, 1)$ ) preserving the symplectic Majorana condition to write down two canonical forms for a single symplectic Majorana spinor corresponding to time-like and null Killing vectors. In section four, we derive the conditions for quarter supersymmetric solutions with time-like Killing vector. In section five, the  $N = 1$  Killing constraints, i. e., the conditions for a time-like quarter supersymmetric solution, are then substituted into the generic Killing spinor equations and the resulting equations are rewritten in the form of constraints on the Kähler base. Section six contains a detailed classification of half-supersymmetric solutions. Our paper ends with two appendices. Appendix A deals with the determination of the linear system obtained from the Killing spinor equations. Appendix B discusses the integrability conditions of the Killing spinor equations. There it is demonstrated that for a given background preserving at least half of the supersymmetry, where at least one of the Killing spinors generates a time-like Killing vector, all of the Einstein, gauge and scalar field equations of motion hold automatically provided that the Bianchi identity is satisfied.

## 2. $N = 2$ supergravity

In this section, we review briefly some aspects of the  $N = 2, D = 5$  gauged supergravity coupled to abelian vector multiplets is [11]. The bosonic action of the theory is

$$S = \frac{1}{16\pi G} \int (-R + 2\chi^2 \mathcal{V}) *1 + Q_{IJ} (dX^I \wedge *dX^J - F^I \wedge *F^J) - \frac{C_{IJK}}{6} F^I \wedge F^J \wedge A^K \quad (2.1)$$

where  $I, J, K$  take values  $1, \dots, n$  and  $F^I = dA^I$  are the two-forms representing gauge field strengths (one of the gauge fields corresponds to the graviphoton). The metric has mostly negative signature. The constants  $C_{IJK}$  are symmetric in  $IJK$  and are not assumed to satisfy the non-linear “adjoint identity” which arises when the scalars lie in a symmetric

space [11]; though we will assume that  $Q_{IJ}$  is invertible, with inverse  $Q^{IJ}$ . The  $X^I$  are scalar fields subject to the constraint

$$\frac{1}{6}C_{IJK}X^IX^JX^K = 1. \quad (2.2)$$

The fields  $X^I$  can thus be regarded as being functions of  $n - 1$  unconstrained scalars  $\phi^r$ . It is convenient to define

$$X_I \equiv \frac{1}{6}C_{IJK}X^JX^K \quad (2.3)$$

so that the condition (2.2) becomes

$$X_I X^I = 1. \quad (2.4)$$

In addition, the coupling  $Q_{IJ}$  depends on the scalars via

$$Q_{IJ} = \frac{9}{2}X_I X_J - \frac{1}{2}C_{IJK}X^K \quad (2.5)$$

so in particular

$$Q_{IJ}X^J = \frac{3}{2}X_I, \quad Q_{IJ}\partial_r X^J = -\frac{3}{2}\partial_r X_I. \quad (2.6)$$

where  $\partial_r$  denotes differentiation with respect to  $\phi^r$ . The scalar potential can be written as

$$\mathcal{V} = 9V_I V_J \left( X^I X^J - \frac{1}{2}Q^{IJ} \right) \quad (2.7)$$

where  $V_I$  are constants.

Bosonic backgrounds are said to be supersymmetric if there exists a spinor  $\epsilon^a$  for which the supersymmetry variations of the gravitino and dilatino vanish in the given background. For the gravitino this requires

$$\left[ \nabla_\mu + \frac{1}{8}X_I (\gamma_\mu F^I{}_{\rho\sigma} \gamma^{\rho\sigma} - 6F^I{}_{\mu\rho} \gamma^\rho) \right] \epsilon^a - \frac{\chi}{2}V_I (X^I \gamma_\mu - 3A^I{}_\mu) \epsilon^{ab} \epsilon^b = 0, \quad (2.8)$$

and for the dilatino it requires

$$\left[ \frac{1}{4} (Q_{IJ} \gamma^{\mu\nu} F^J{}_{\mu\nu} + 3\gamma^\mu \nabla_\mu X_I) \epsilon^a - \frac{3\chi}{2}V_I \epsilon^{ab} \epsilon^b \right] \partial_r X^I = 0. \quad (2.9)$$

The Einstein equation derived from (2.1) is given by

$$R_{\alpha\beta} + Q_{IJ} \left( F^I{}_{\alpha\lambda} F^J{}_{\beta}{}^\lambda - \nabla_\alpha X^I \nabla_\beta X^J - \frac{1}{6}g_{\alpha\beta} F^I{}_{\mu\nu} F^{J\mu\nu} \right) - \frac{2}{3}g_{\alpha\beta} \chi^2 \mathcal{V} = 0. \quad (2.10)$$

The Maxwell equations (varying  $A^I$ ) are

$$d(Q_{IJ} \star F^J) = -\frac{1}{4}C_{IJK} F^J \wedge F^K. \quad (2.11)$$

The scalar equations (varying  $\phi^r$ ) are

$$\left[ -d(\star dX_I) + \left( X_M X^P C_{NPI} - \frac{1}{6}C_{MNI} \right) (F^M \wedge \star F^N - dX^M \wedge \star dX^N) - \frac{3}{2}\chi^2 V_M V_N Q^{ML} Q^{NP} C_{LPI} \text{dvol} \right] \partial_r X^I = 0. \quad (2.12)$$

If a quantity  $L_I$  satisfies  $L_I \partial_r X^I = 0$ , then there must be a function  $\Upsilon$  such that  $L_I = \Upsilon X_I$ . This implies that the dilatino equation (2.9) can be simplified to

$$F^I{}_{\mu\nu} \gamma^{\mu\nu} \epsilon^a = (X^I X_J F^J{}_{\mu\nu} \gamma^{\mu\nu} + 2\gamma^\mu \nabla_\mu X^I) \epsilon^a - 4\chi V_J \left( X^I X^J - \frac{3}{2} Q^{IJ} \right) \epsilon^{ab} \epsilon^b \quad (2.13)$$

and the scalar equation can be written as

$$\begin{aligned} & -d(\star dX_I) + \left( \frac{1}{6} C_{MNI} - \frac{1}{2} X_I C_{MNI} X^J \right) dX^M \wedge \star dX^N \\ & + \left( X_M X^P C_{NPI} - \frac{1}{6} C_{MNI} - 6X_I X_M X_N + \frac{1}{6} X_I C_{MNI} X^J \right) F^M \wedge \star F^N \\ & + 3\chi^2 V_M V_N \left( \frac{1}{2} Q^{ML} Q^{NP} C_{LPI} + X_I Q^{MN} - 2X_I X^M X^N \right) \text{dvol} = 0. \end{aligned} \quad (2.14)$$

### 3. Spinors in five dimensions

Following [23–25], we write spinors in five dimensions as forms on  $\Lambda^*(\mathbb{R}^2) \otimes \mathbb{C}$ . We represent a generic spinor  $\eta$  in the form

$$\eta = \lambda 1 + \mu^i e^i + \sigma e^{12}, \quad (3.1)$$

where  $e^1, e^2$  are 1-forms on  $\mathbb{R}^2$ ,  $e^{12} = e^1 \wedge e^2$  and  $\lambda, \mu^i$  and  $\sigma$  are complex functions.

The action of  $\gamma$ -matrices on these forms is given by

$$\gamma_i = i(e^i \wedge + i_{e^i}), \quad \gamma_{i+2} = -e^i \wedge + i_{e^i}. \quad (3.2)$$

We define  $\gamma_0$  by  $\gamma_0 = \gamma_{1234}$ . This satisfies

$$\gamma_0 1 = 1, \quad \gamma_0 e^{12} = e^{12}, \quad \gamma_0 e^i = -e^i. \quad (3.3)$$

The charge conjugation operator  $C$  is defined by

$$C 1 = -e^{12}, \quad C e^{12} = 1, \quad C e^i = -\epsilon_{ij} e^j \quad (3.4)$$

where  $\epsilon_{ij} = \epsilon^{ij}$  is antisymmetric with  $\epsilon_{12} = 1$ .

The Killing spinors  $\epsilon^a$  of the theory satisfy a symplectic Majorana constraint which is

$$(\epsilon^a)^* = \epsilon^{ab} \gamma_0 C \epsilon^b \quad (3.5)$$

so if one writes

$$\epsilon^1 = \lambda 1 + \mu^i e^i + \sigma e^{12}, \quad (3.6)$$

then  $\epsilon^2$  is fixed via

$$\epsilon^2 = -\sigma^* 1 - \epsilon_{ij} (\mu^i)^* e^j + \lambda^* e^{12}. \quad (3.7)$$

We note the useful identity

$$(\gamma_\mu)^* = -\gamma_0 C \gamma_\mu \gamma_0 C. \quad (3.8)$$

It will be particularly useful in our work to complexify the gamma operators. Therefore we write

$$\begin{aligned} \gamma_p &= \frac{1}{\sqrt{2}} (\gamma_p - i\gamma_{p+2}) = \sqrt{2} i e^p \wedge \\ \gamma_{\bar{p}} &= \frac{1}{\sqrt{2}} (\gamma_p + i\gamma_{p+2}) = \sqrt{2} i i e^p. \end{aligned} \quad (3.9)$$

### 3.1 Gauge transformations and $N = 1$ spinors

There are two types of gauge transformation that preserve the symplectic Majorana condition (3.5). First, we have the  $U(1)$  gauge transformations described by

$$\begin{pmatrix} \epsilon^1 \\ \epsilon^2 \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \epsilon^1 \\ \epsilon^2 \end{pmatrix} \quad (3.10)$$

and there are also  $\text{Spin}(4, 1)$  gauge transformations of the form

$$\epsilon^a \rightarrow e^{\frac{1}{2}f^{\mu\nu}\gamma_{\mu\nu}}\epsilon^a, \quad (3.11)$$

for real functions  $f^{\mu\nu}$ .

Note in particular that  $\frac{1}{2}(\gamma_{12} + \gamma_{34})$ ,  $\frac{1}{2}(\gamma_{13} - \gamma_{24})$  and  $\frac{1}{2}(\gamma_{14} + \gamma_{23})$  generate a  $SU(2)$  which leaves 1 and  $e^{12}$  invariant and acts on  $e^1, e^2$ ; whereas  $\frac{1}{2}(\gamma_{12} - \gamma_{34})$ ,  $\frac{1}{2}(\gamma_{13} + \gamma_{24})$  and  $\frac{1}{2}(\gamma_{14} - \gamma_{23})$  generate another  $SU(2)$  which leaves the  $e^i$  invariant but acts on 1 and  $e^{12}$ . In addition,  $\gamma_{03}$  generates a  $SO(1, 1)$  which acts (simultaneously) on 1,  $e^1$  and  $e^2, e^{12}$ , whereas  $\gamma_{04}$  generates another  $SO(1, 1)$  which acts (simultaneously) on 1,  $e^2$  and  $e^1, e^{12}$ .

Therefore, for a single symplectic Majorana spinor, one can always use  $\text{Spin}(4, 1)$  gauge transformations to write

$$\epsilon^1 = f1, \quad \epsilon^2 = fe^{12}, \quad (3.12)$$

or

$$\epsilon^1 = fe^1, \quad \epsilon^2 = -fe^2, \quad (3.13)$$

or

$$\epsilon^1 = f(1 + e^1), \quad \epsilon^2 = f(-e^2 + e^{12}), \quad (3.14)$$

for some real function  $f$ . However, under the transformation

$$\begin{aligned} \epsilon^a &\rightarrow \gamma_1 \epsilon^a, \\ \gamma_\mu &\rightarrow -\gamma_1 \gamma_\mu \gamma_1, \\ C &\rightarrow -\gamma_1 C \gamma_1, \end{aligned} \quad (3.15)$$

the spinor in (3.13) transforms as

$$\epsilon^1 \rightarrow if1, \quad \epsilon^2 \rightarrow -ife^{12} \quad (3.16)$$

and

$$\gamma_0 \rightarrow -\gamma_0, \quad \gamma_1 \rightarrow \gamma_1, \quad \gamma_2 \rightarrow -\gamma_2, \quad \gamma_3 \rightarrow -\gamma_3, \quad \gamma_4 \rightarrow -\gamma_4 \quad (3.17)$$

and  $C$  is unchanged. This transformation corresponds to reflections in the 0, 2, 3, 4 directions. Moreover, the spinor in (3.16) is equivalent to that in (3.12) under a  $SU(2)$  gauge transformation. The spinors corresponding to (3.12) and (3.13) are therefore equivalent under these transformations. Hence, for a single spinor, one need only consider the cases (3.12) and (3.14).

### 3.2 Differential forms from spinors

In order to define differential forms, we first define a Hermitian inner product on  $\Lambda^*\mathbb{R}^2 \otimes \mathbb{C}$  by

$$\langle z^0 1 + z^1 e^1 + z^2 e^2 + z^3 e^{12}, w^0 1 + w^1 e^1 + w^2 e^2 + w^3 e^{12} \rangle = \sum_{\alpha=0}^3 \bar{z}^\alpha w^\alpha . \quad (3.18)$$

Then Spin(4, 1) gauge-invariant  $k$ -forms are obtained from spinors  $\epsilon, \eta$  via

$$\alpha(\epsilon, \eta)_{\mu_1, \dots, \mu_k} = -\langle C\epsilon^*, \gamma_{\mu_1, \dots, \mu_k} \eta \rangle. \quad (3.19)$$

In particular, for the generic Majorana spinor given in (3.6) and (3.7) one finds

$$\alpha(\epsilon^a, \epsilon^b)_{\mu_1, \dots, \mu_k} = \langle \epsilon^{ac} \gamma_0 \epsilon^c, \gamma_{\mu_1, \dots, \mu_k} \epsilon^b \rangle. \quad (3.20)$$

The scalars are then given by

$$\alpha(\epsilon^a, \epsilon^b) = \epsilon^{ab} (|\sigma|^2 + |\lambda|^2 - |\mu_1|^2 - |\mu_2|^2). \quad (3.21)$$

Hence, by comparing with [18], it is clear that the spinor given in (3.12) corresponds to the time-like class of solutions, whereas that in (3.14) is in the null class of solutions. With a slight abuse of notation, we shall refer to the corresponding Killing spinors as being either time-like or null.

### 3.3 Canonical $N = 2$ spinors

We will now assume that there are two linearly independent symplectic Majorana Killing spinors  $\epsilon^a, \eta^a$ , where  $\epsilon^a$  is time-like. In appendix B it is demonstrated that the existence of such spinors is sufficient to ensure that the scalar, gauge and Einstein equations of motion hold automatically from the integrability conditions, provided one assumes that the Bianchi identities are satisfied. So the only equations which must be solved are the Killing spinor equations together with the Bianchi identity.

From the previous reasoning, we can take  $\epsilon^a$  to have the canonical form.

$$\epsilon^1 = f, \quad \epsilon^2 = f e^{12} \quad (3.22)$$

for  $f \in \mathbb{R}$ . Next consider  $\eta^a$  given by

$$\eta^1 = \lambda 1 + \mu^i e^i + \sigma e^{12}, \quad (3.23)$$

$$\eta^2 = -\sigma^* 1 - \epsilon_{ij} (\mu^i)^* e^j + \lambda^* e^{12} \quad (3.24)$$

for complex  $\lambda, \mu_i, \sigma$ . It is possible to simplify  $\eta^a$  a little using gauge transformations which leave  $\epsilon^a$  invariant. In particular, by using an appropriate SU(2) transformation, one could for example set  $\mu^2 = 0$  with  $\mu^1 \in \mathbb{R}$ . However, we will not make this gauge choice.



### 3.4 The 1/4 supersymmetric time-like solution

In this section we obtain the time-like solutions preserving a quarter of the supersymmetry using the spinorial geometry method. These solutions were derived in [21, 22]. In order to obtain 1/4 supersymmetric solutions with time-like Killing spinor, it suffices to consider the equations (A.1)–(A.12) and set  $\sigma = \mu^p = 0$  and  $\lambda = f$ . Then from the dilatino equation, we find

$$F^I{}_m{}^m = X^I H_m{}^m - \partial_0 X^I, \quad (3.25)$$

$$F^I{}_{0n} = X^I H_{0n} - \partial_n X^I, \quad (3.26)$$

$$(F^I{}_{mn} - X^I H_{mn}) \epsilon^{mn} = 2\chi V_J (X^I X^J - \frac{3}{2} Q^{IJ}), \quad (3.27)$$

whereas from the gravitino equation we find

$$\frac{1}{f} \partial_0 f - \frac{1}{4} (2\omega_{0,m}{}^m + H_m{}^m) = 0, \quad (3.28)$$

$$\omega_{0,0n} + H_{0n} = 0, \quad (3.29)$$

$$(H_{mn} + 2\omega_{0,mn}) \epsilon^{mn} + 2\chi V_I (X^I - 3A^I{}_0) = 0, \quad (3.30)$$

$$\frac{1}{f} \partial_p f - \frac{1}{4} (2\omega_{p,m}{}^m - 3H_{0p}) = 0, \quad (3.31)$$

$$H_{p\bar{q}} - \frac{1}{3} (H_m{}^m \delta_{p\bar{q}} - 2\omega_{p,0\bar{q}}) = 0, \quad (3.32)$$

$$-\omega_{p,\bar{m}\bar{n}} \epsilon^{\bar{m}\bar{n}} + H_{0\bar{n}} \epsilon^{\bar{n}}{}_p + 3\chi V_I A^I{}_p = 0, \quad (3.33)$$

$$\frac{1}{f} \partial_{\bar{p}} f - \frac{1}{4} (2\omega_{\bar{p},m}{}^m - H_{0\bar{p}}) = 0, \quad (3.34)$$

$$\omega_{p,0q} + \left( \frac{1}{4} H_{mn} \epsilon^{mn} - \chi V_I X^I \right) \epsilon_{pq} = 0, \quad (3.35)$$

$$3\chi V_I A^I{}_{\bar{p}} - \omega_{\bar{p},\bar{m}\bar{n}} \epsilon^{\bar{m}\bar{n}} = 0. \quad (3.36)$$

To analyze this linear system, we will first consider the gravitino equations. Note that (3.28) implies that

$$\partial_0 f = 0, \quad (3.37)$$

and

$$H_m{}^m = -2\omega_{0,m}{}^m. \quad (3.38)$$

Next, consider (3.29), (3.31) and (3.34). These imply

$$H_{0p} = -\frac{2}{f} \partial_p f, \quad (3.39)$$

$$\omega_{0,0p} = \frac{2}{f} \partial_p f, \quad (3.40)$$

$$\omega_{p,m}{}^m = -\frac{1}{f} \partial_p f. \quad (3.41)$$

From (3.30) and (3.35) we find

$$\omega_{(p,q)0} = 0, \quad (3.42)$$

and

$$\omega_{[\bar{m},0]\bar{n}}\epsilon^{\bar{m}\bar{n}} + \frac{3\chi}{2}V_I(A^I_0 - X^I) = 0. \quad (3.43)$$

From (3.32) we find

$$\omega_{(p,\bar{q})0} = 0 \quad (3.44)$$

and from (3.33) and (3.36) we obtain

$$\omega_{p,mn}\epsilon^{mn} - \omega_{p,\bar{m}\bar{n}}\epsilon^{\bar{m}\bar{n}} - \frac{2}{f}\epsilon^{\bar{n}}{}_p\partial_{\bar{n}}f = 0 \quad (3.45)$$

and

$$3\chi V_I A^I_p = \omega_{p,mn}\epsilon^{mn}. \quad (3.46)$$

Hence, to summarize, we obtain the following purely geometric constraints

$$\begin{aligned} \partial_0 f &= 0, \\ \omega_{0,0p} &= 2\frac{\partial_p f}{f}, \\ \omega_{(p,q)0} &= 0, \\ \omega_{(p,\bar{q})0} &= 0, \end{aligned} \quad (3.47)$$

together with

$$\begin{aligned} \omega_{p,m}{}^m + \frac{\partial_p f}{f} &= 0, \\ \omega_{p,mn}\epsilon^{mn} - \omega_{p,\bar{m}\bar{n}}\epsilon^{\bar{m}\bar{n}} - \frac{2}{f}\epsilon^{\bar{n}}{}_p\partial_{\bar{n}}f &= 0. \end{aligned} \quad (3.48)$$

It is straightforward to show that the constraints (3.47) are the necessary and sufficient conditions for the 1-form

$$\kappa = f^2 \mathbf{e}^0 \quad (3.49)$$

to define a Killing vector  $V$ . This form is, as expected, the 1-form spinor bilinear which is obtained from  $\epsilon^a$ . Note that this Killing vector satisfies

$$\mathcal{L}_V \mathbf{e}^0 = 0. \quad (3.50)$$

In fact, one can choose a gauge in which the Lie derivative of the vielbein with respect to  $V$  vanishes. To see this, note that

$$\mathcal{L}_V \mathbf{e}^p = f^2(\omega_{0,\bar{p}q} - \omega_{q,\bar{p}0})\mathbf{e}^q + f^2\epsilon_{\bar{p}\bar{q}}\omega_{[0,\bar{m}]\bar{n}}\epsilon^{\bar{m}\bar{n}}\mathbf{e}^{\bar{q}}. \quad (3.51)$$

From (3.43) we note that  $\omega_{[0,\bar{m}]\bar{n}}\epsilon^{\bar{m}\bar{n}} \in \mathbb{R}$ . Without loss of generality we can make a  $U(1)$  gauge transformation in order to set

$$V_I A^I_0 = V_I X^I. \quad (3.52)$$

This gauge transformation alters the form of the Killing spinors via the transformation given in (3.10). However, the Killing spinors can be restored to their original form by

making a  $U(1) \in SU(2) \subset Spin(4, 1)$  gauge transformation generated by  $\gamma_{12} - \gamma_{34}$ . Working in this gauge, (3.43) implies that  $\omega_{[0, \bar{m}] \bar{n}} \epsilon^{\bar{m} \bar{n}} = 0$  and hence

$$\mathcal{L}_V \mathbf{e}^p = A^p_q \mathbf{e}^q \tag{3.53}$$

where the constraints in (3.32), (3.38) and (3.47) imply that  $A \in su(2)$ . By making a further  $SU(2) \subset Spin(4, 1)$  gauge transformation generated by  $\gamma_{12} + \gamma_{34}, \gamma_{13} - \gamma_{24}, \gamma_{14} + \gamma_{23}$  which leaves  $1, e^{12}$  invariant and maps  $e^p \rightarrow X^p_q e^q$  for  $X \in SU(2)$ , we can without loss of generality take  $A = 0$ . In this basis the vielbein is time-independent.

Equation (3.48) also has a simple geometric interpretation. First note that the only  $U(1)$  gauge-invariant 2-form which can be obtained from  $\epsilon$  is the real part of the 2-form

$$\alpha_{\mu\nu}(\epsilon^1, \epsilon^1) = -\langle Cf.1, \gamma_{\mu\nu} f1 \rangle = f^2 \langle e^{12}, \gamma_{\mu\nu} 1 \rangle. \tag{3.54}$$

The real part of this form (denoted by  $J$ ) is then given by

$$J_{pq} = -f^2 \epsilon_{pq}, \quad J_{\bar{p}\bar{q}} = -f^2 \epsilon_{\bar{p}\bar{q}}. \tag{3.55}$$

It is convenient to make a conformal rescaling of the complexified basis and define

$$\mathbf{e}^p = f^{-1} \hat{\mathbf{e}}^p, \quad \mathbf{e}^{\bar{p}} = f^{-1} \hat{\mathbf{e}}^{\bar{p}}. \tag{3.56}$$

We shall refer to the 4-manifold with metric

$$ds_4^2 = 2 \left( \hat{\mathbf{e}}^1 \hat{\mathbf{e}}^{\bar{1}} + \hat{\mathbf{e}}^2 \hat{\mathbf{e}}^{\bar{2}} \right) \tag{3.57}$$

as the base space  $B$ . Then it is clear that  $J$  defines an almost complex structure on this 4-manifold. In fact, (3.48) implies that  $J$  is covariantly constant with respect to the Levi-civita connection of the base manifold, and hence  $B$  is a Kähler manifold (as expected) with Kähler form  $J$ .

Also, from (3.36) we have

$$3\chi V_I A^I_p = \omega_{p, mn} \epsilon^{mn}. \tag{3.58}$$

We remark that (3.58) implies that

$$\mathcal{P} = 3\chi V_I (A^I_p \mathbf{e}^p + A^I_{\bar{p}} \mathbf{e}^{\bar{p}}). \tag{3.59}$$

where  $\mathcal{P}$  is (locally) the potential for the Ricci form of the Kähler base  $B$ .<sup>1</sup>

The remaining constraints on the  $H$ -flux are then

$$H_{0p} = -2 \frac{\partial_p f}{f}, \tag{3.60}$$

$$H_{mn} \epsilon^{mn} = -2\omega_{0, mn} \epsilon^{mn} + 4\chi V_I X^I, \tag{3.61}$$

$$H_{p\bar{q}} = -\frac{2}{3} (\omega_{0, m}{}^m \delta_{p\bar{q}} + \omega_{0, p\bar{q}}). \tag{3.62}$$

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<sup>1</sup>If we are in the ungauged theory with  $\chi = 0$ , then the vanishing of  $\omega_{p, mn}$  is then sufficient to imply that the base  $B$  is hyper-Kähler. But we shall take  $\chi \neq 0$  throughout.

Finally, we substitute these constraints into the dilatino equations. From (3.25)–(3.27), we find

$$\partial_0 X^I = 0, \quad (3.63)$$

$$F^I{}_m{}^m = -2\omega_{0,m}{}^m X^I, \quad (3.64)$$

$$F^I{}_{0n} = -\frac{1}{f^2} \partial_n (f^2 X^I), \quad (3.65)$$

$$(F^I{}_{mn} + 2X^I \omega_{0,mn}) \epsilon^{mn} = 3\chi V_J (2X^I X^J - Q^{IJ}). \quad (3.66)$$

#### 4. Killing spinor in $N = 1$ background

In this section, we substitute the constraints obtained in the previous section back into the generic Killing spinor equation (A.1)–(A.12) and simplify as much as possible. We find from the dilatino equation:

$$\mu^m \partial_m X^I = -\sqrt{2} \chi V_J \left( X^I X^J - \frac{3}{2} Q^{IJ} \right) \text{Im } \sigma, \quad (4.1)$$

$$\mu^m \left[ F^I{}_{m\bar{q}} + \frac{2}{3} (\omega_{0,p}{}^p \delta_{m\bar{q}} + \omega_{0,m\bar{q}}) X^I \right] = \chi V_J \left( X^I X^J - \frac{3}{2} Q^{IJ} \right) \epsilon_{\bar{m}\bar{q}} (\mu^m)^*, \quad (4.2)$$

$$\mu^m \epsilon^{\bar{n}}{}_m \partial_{\bar{n}} X^I = \sqrt{2} \chi V_J \left( X^I X^J - \frac{3}{2} Q^{IJ} \right) \text{Im } \lambda. \quad (4.3)$$

And from the gravitino equations we find

$$\partial_0 \lambda = 2i \left( \sqrt{2} \frac{\mu^m}{f} \partial_m f - \chi V_I X^I \text{Im } \sigma \right), \quad (4.4)$$

$$\partial_0 \sigma = 2i \left( \sqrt{2} \frac{\mu^m}{f} \epsilon^{\bar{n}}{}_m \partial_{\bar{n}} f + \chi V_I X^I \text{Im } \lambda \right), \quad (4.5)$$

$$\partial_0 \mu_{\bar{q}} = -\frac{2}{3} \mu^m (2\omega_{0,m\bar{q}} - \omega_{0,p}{}^p \delta_{m\bar{q}}) + 2\chi V_I X^I \epsilon_{\bar{m}\bar{q}} (\mu^m)^*, \quad (4.6)$$

and

$$\partial_{\bar{p}} \left( \frac{\sigma}{f} \right) = \frac{i}{f} \left( -\sqrt{2} \mu_{\bar{p}} \left( \omega_{0,\bar{r}\bar{n}} \epsilon^{\bar{r}\bar{n}} - \frac{3\chi V_I X^I}{2} \right) + \omega_{\bar{p},\bar{m}\bar{n}} \epsilon^{\bar{m}\bar{n}} \text{Im } \lambda \right), \quad (4.7)$$

$$\partial_p \left( \frac{\lambda}{f} \right) = \frac{i}{f} \left( \sqrt{2} \mu^m \left( 2\omega_{0,pm} - \frac{3\chi V_I X^I}{2} \epsilon_{pm} \right) - \omega_{p,mn} \epsilon^{mn} \text{Im } \sigma \right), \quad (4.8)$$

$$\partial_p \left( \frac{\sigma}{f} \right) = \frac{2i\sqrt{2}\mu^m}{3f} (\omega_{0,p\bar{n}} \epsilon^{\bar{n}}{}_m + \omega_{0,n}{}^n \epsilon_{pm}) - \frac{i\chi V_I X^I}{\sqrt{2}f} (\mu_{\bar{p}})^* + \frac{i}{f} \omega_{p,mn} \epsilon^{mn} \text{Im } \lambda, \quad (4.9)$$

$$\partial_{\bar{p}} \left( \frac{\lambda}{f} \right) = \frac{2i\sqrt{2}\mu^m}{3f} (\omega_{0,\bar{p}m} - \omega_{0,n}{}^n \delta_{\bar{p}m}) + \frac{i\chi V_I X^I}{\sqrt{2}f} \epsilon_{\bar{p}\bar{m}} (\mu^m)^* - \frac{i}{f} \omega_{\bar{p},\bar{m}\bar{n}} \epsilon^{\bar{m}\bar{n}} \text{Im } \sigma, \quad (4.10)$$

$$\begin{aligned} \partial_p \mu_{\bar{q}} &= (\mu^m)^* \left( \omega_{p,\bar{m}\bar{q}} + \epsilon^{\bar{n}}{}_p \epsilon_{\bar{m}\bar{q}} \frac{\partial_{\bar{n}} f}{f} \right) - \mu^m \left( 2\delta_{m\bar{q}} \frac{\partial_p f}{f} - \delta_{p\bar{q}} \frac{\partial_m f}{f} + \omega_{p,m\bar{q}} \right) \\ &\quad - \sqrt{2} \chi V_I X^I \delta_{p\bar{q}} \text{Im } \sigma, \end{aligned} \quad (4.11)$$

$$\begin{aligned} \partial_{\bar{p}} \mu_{\bar{q}} &= -\mu^m \left( \delta_{\bar{q}m} \frac{\partial_{\bar{p}} f}{f} - \epsilon^{\bar{n}}{}_m \epsilon_{\bar{p}\bar{q}} \frac{\partial_{\bar{n}} f}{f} + \omega_{\bar{p},m\bar{q}} \right) + \omega_{\bar{p},\bar{m}\bar{q}} (\mu^m)^* \\ &\quad + \sqrt{2} \chi V_I X^I \epsilon_{\bar{p}\bar{q}} \text{Im } \lambda. \end{aligned} \quad (4.12)$$

Note that these equations admit a solution of the form  $\lambda = \rho_1 f$ ,  $\sigma = \rho_2 f$ ,  $\mu^p = 0$  with  $\rho_1, \rho_2$  real constants, and no additional constraints on the fluxes or geometry. Hence we observe that the generic time-like solution preserves 1/4 supersymmetry. More generally, if  $\eta^1, \eta^2$  are symplectic Majorana Killing spinors, then so are

$$(\eta^1)' = \eta^2, \quad (\eta^2)' = -\eta^1 \tag{4.13}$$

which is just a special case of (3.10) with  $\theta = \frac{\pi}{2}$ . In particular, the equations computed above are invariant under the transformations

$$\begin{aligned} \lambda &\rightarrow -\sigma^*, \\ \sigma &\rightarrow \lambda^*, \\ \mu^p &\rightarrow \epsilon^p_{\bar{q}}(\mu^{\bar{q}})^* \end{aligned} \tag{4.14}$$

therefore it is clear that the Killing spinors arise in pairs.

#### 4.1 Solutions with $\mu^p = 0$

Suppose we consider the case when  $\mu^p = 0$ . Then, assuming that  $V_I X^I \neq 0$ , we find from (4.11) and (4.12) that

$$\text{Im } \sigma = \text{Im } \lambda = 0. \tag{4.15}$$

If, however,  $V_I X^I = 0$  then (4.1) and (4.3) again imply (4.15), as we assume that not all of the  $V_I$  vanish.

Hence, from (4.4), (4.5), (4.8) and (4.9) it follows directly that  $\lambda = \rho_1 f$ ,  $\sigma = \rho_2 f$ ,  $\mu^p = 0$  with  $\rho_1, \rho_2$  real constants. The solution is therefore only 1/4-supersymmetric. Thus, to find new solutions with enhanced supersymmetry, one must take  $\mu^p \neq 0$ ; henceforth we shall assume that  $\mu^p \neq 0$ .

#### 4.2 Constraints on the base space

It will be particularly useful to rewrite the equations (4.1)–(4.12) in terms of constraints on the Kähler base. Throughout this section, unless stated otherwise, tensor indices are evaluated with respect to the 4-dimensional complex basis  $\hat{e}^p, \hat{e}^{\bar{p}}$ ; so we shall drop the  $\hat{\cdot}$  from all expressions. It is convenient to define a real vector field  $K$  on the Kähler base as follows

$$K^p = i f^2 \mu^p, \quad K^{\bar{p}} = -i f^2 (\mu^p)^*. \tag{4.16}$$

In order to rewrite the constraints, we define a time co-ordinate  $t$  so that the Killing vector field associated with the Killing spinor  $\epsilon^a$  is

$$V = \frac{\partial}{\partial t} \tag{4.17}$$

and set

$$e^0 = f^2(dt + \Omega) \tag{4.18}$$

where  $\Omega$  is a 1-form defined on the Kähler base.

Then (4.11) is equivalent to

$$\nabla_p K_{\bar{q}} = \frac{1}{\sqrt{2}f} \partial_t \lambda \delta_{p\bar{q}} + \Omega_p \partial_t K_{\bar{q}} \quad (4.19)$$

where here  $\nabla$  denotes the Levi-civita connection of the Kähler base metric given in (3.57). Also, (4.12) can be rewritten as

$$\nabla_p K_q = \frac{1}{\sqrt{2}f} \partial_t \sigma^* \epsilon_{pq} + \Omega_p \partial_t K_q. \quad (4.20)$$

It is also useful to define

$$Z = i_K J. \quad (4.21)$$

It is then straightforward to show that

$$\begin{aligned} \nabla_p Z_{\bar{q}} &= \frac{1}{\sqrt{2}f} \partial_t \sigma^* \delta_{p\bar{q}} + \Omega_p \partial_t Z_{\bar{q}}, \\ \nabla_p Z_q &= -\frac{1}{\sqrt{2}f} \partial_t \lambda \epsilon_{pq} + \Omega_p \partial_t Z_q. \end{aligned} \quad (4.22)$$

The commutator is given by

$$\begin{aligned} [K, Z]^p &= -i \frac{\sqrt{2}}{f} (K^p \partial_t (\text{Im } \sigma) + Z^p \partial_t (\text{Im } \lambda)) + (i_K \Omega) \partial_t Z^p - (i_Z \Omega) \partial_t K^p, \\ [K, Z]^{\bar{p}} &= i \frac{\sqrt{2}}{f} (K^{\bar{p}} \partial_t (\text{Im } \sigma) + Z^{\bar{p}} \partial_t (\text{Im } \lambda)) + (i_K \Omega) \partial_t Z^{\bar{p}} - (i_Z \Omega) \partial_t K^{\bar{p}}, \end{aligned} \quad (4.23)$$

Next, (4.1) and (4.3) are equivalent to

$$K^p \nabla_p X^I = -\sqrt{2} i \chi f V_J \left( X^I X^J - \frac{3}{2} Q^{IJ} \right) \text{Im } \sigma, \quad (4.24)$$

$$Z^{\bar{p}} \nabla_{\bar{p}} X^I = \sqrt{2} i \chi f V_J \left( X^I X^J - \frac{3}{2} Q^{IJ} \right) \text{Im } \lambda. \quad (4.25)$$

These equations simply imply that

$$\mathcal{L}_K X^I = \mathcal{L}_Z X^I = 0. \quad (4.26)$$

In addition, (4.4) and (4.5) can be rewritten as

$$\partial_t \lambda = 2 \left( \sqrt{2} K^p \nabla_p f - i \chi f^2 V_I X^I \text{Im } \sigma \right), \quad (4.27)$$

and

$$\partial_t \sigma = 2 \left( \sqrt{2} Z^{\bar{p}} \nabla_{\bar{p}} f + i \chi f^2 V_I X^I \text{Im } \lambda \right). \quad (4.28)$$

In order to simplify the remainder of the equations, observe that for indices  $\mu, \nu \neq 0$ ,

$$\omega_{0, \mu\nu} = -\frac{1}{2} f^4 (d\hat{\Omega})_{\mu\nu} \quad (4.29)$$

and

$$\omega_{p,qr} = -f\hat{\omega}_{p,qr} \quad (4.30)$$

where on the l.h.s. of (4.29) and (4.30), spatial indices are taken with respect to the original five-dimensional basis, whereas on the r.h.s., they are taken with respect to the conformally rescaled Kähler basis; and  $\hat{\omega}$  denotes the spin connection of the Kähler base space. From henceforth, the hat will be dropped, and we will work solely on the Kähler base space.

Then (4.6) is equivalent to

$$\partial_t K_{\bar{q}} = \frac{1}{3}f^6 K^p (2d\Omega_{p\bar{q}} - d\Omega_m{}^m \delta_{p\bar{q}}) + 2\chi f^2 V_I X^I Z_{\bar{q}} \quad (4.31)$$

and using this, (4.2) can be rewritten as

$$K^p F^I{}_{p\bar{q}} = f^2 X^I K^p d\Omega_{p\bar{q}} + \frac{3\chi}{f^2} \left( X^I X^J - \frac{1}{2} Q^{IJ} \right) V_J Z_{\bar{q}} - \frac{1}{f^4} \partial_t K_{\bar{q}} X^I. \quad (4.32)$$

It is also useful to rewrite (3.66) as

$$F^I{}_{pq} = f^2 X^I d\Omega_{pq} + \frac{3\chi}{f^2} \left( X^I X^J - \frac{1}{2} Q^{IJ} \right) V_J \epsilon_{pq}. \quad (4.33)$$

In addition, (4.8), (4.9), (4.10) and (4.7) can be rewritten as

$$\nabla_p \left( \text{Re} \frac{\lambda}{f} \right) - \frac{1}{\sqrt{2}} (i_K d\Omega)_p + \frac{1}{\sqrt{2}f^6} \partial_t K_p - \frac{\Omega_p}{f} \partial_t \text{Re} \lambda = 0, \quad (4.34)$$

$$\nabla_p \left( \text{Re} \frac{\sigma}{f} \right) - \frac{1}{\sqrt{2}} (i_Z d\Omega)_p + \frac{1}{\sqrt{2}f^6} \partial_t Z_p - \frac{\Omega_p}{f} \partial_t \text{Re} \sigma = 0, \quad (4.35)$$

$$\begin{aligned} \nabla_p \left( \text{Im} \frac{\lambda}{f} \right) &= \frac{i}{2\sqrt{2}} \left( d\Omega_{mn} \epsilon^{mn} + \frac{2\chi}{f^4} V_I X^I \right) Z_p - i \frac{\sqrt{2}}{6} (K^{\bar{q}} d\Omega_{p\bar{q}} + K_p d\Omega_q{}^q) \\ &\quad + \frac{1}{f} \omega_{p,mn} \epsilon^{mn} \text{Im} \sigma + \frac{\Omega_p}{f} \partial_t \text{Im} \lambda, \end{aligned} \quad (4.36)$$

$$\begin{aligned} \nabla_p \left( \text{Im} \frac{\sigma}{f} \right) &= \frac{i}{2\sqrt{2}} \left( d\Omega_{mn} \epsilon^{mn} + \frac{2}{f^4} \chi V_I X^I \right) K_p + \frac{i\sqrt{2}}{6} (Z^{\bar{q}} d\Omega_{p\bar{q}} + Z_p d\Omega_q{}^q) \\ &\quad - \frac{1}{f} \omega_{p,mn} \epsilon^{mn} \text{Im} \lambda + \frac{\Omega_p}{f} \partial_t \text{Im} \sigma. \end{aligned} \quad (4.37)$$

## 5. Half supersymmetric solutions

Suppose that the solution preserves exactly four of the supersymmetries. Then the four linearly independent Killing spinors are  $\epsilon^1, \epsilon^2, \eta^1, \eta^2$  and

$$\begin{aligned} (\epsilon^1)' &= \epsilon^2, & (\epsilon^2)' &= -\epsilon^1, \\ (\eta^1)' &= \eta^2, & (\eta^2)' &= -\eta^1. \end{aligned} \quad (5.1)$$

As all of the scalars, gauge field strengths and components of the spin connection are  $t$ -independent, it follows that  $\partial_t \eta^1, \partial_t \eta^2$  is also a Killing spinor. As the solution is exactly half-supersymmetric, it follows that there must be real constants  $c_1, c_2, c_3, c_4$  such that

$$\partial_t \eta^1 = c_1 \eta^1 + c_2 \eta^2 + c_3 \epsilon^1 + c_4 \epsilon^2 \quad (5.2)$$

or equivalently

$$\begin{aligned}\partial_t \lambda &= c_1 \lambda - c_2 \sigma^* + c_3 f, \\ \partial_t \sigma &= c_1 \sigma + c_2 \lambda^* + c_4 f, \\ \partial_t \mu^p &= c_1 \mu^p - c_2 \epsilon_{\bar{q}}^p (\mu^q)^*.\end{aligned}\tag{5.3}$$

On substituting these constraints into (4.6) we find that

$$\frac{1}{f^2} (c_1 \mu_{\bar{q}} + c_2 \epsilon_{\bar{q}\bar{n}} (\mu^n)^*) + \frac{2}{3} \mu^m (2\omega_{0,m\bar{q}} - \omega_{0,p}{}^p \delta_{m\bar{q}}) - 2\chi V_I X^I \epsilon_{\bar{m}\bar{q}} (\mu^m)^* = 0.\tag{5.4}$$

Contracting this expression with  $(\mu^q)^*$  we find that

$$(\mu^q)^* \left( \frac{c_1}{f^2} \mu_{\bar{q}} + \frac{2}{3} \mu^m (2\omega_{0,m\bar{q}} - \omega_{0,p}{}^p \delta_{m\bar{q}}) \right) = 0.\tag{5.5}$$

The real part of this expression implies that  $c_1 = 0$ .

Suppose now that  $c_2 \neq 0$ . From (5.3) we find that

$$\partial_t \mu^p = -c_2 \epsilon_{\bar{q}}^p (\mu^q)^*\tag{5.6}$$

and hence

$$\mu^p = \alpha^p \cos(c_2 t) + \epsilon_{\bar{q}}^p (\alpha^q)^* \sin(c_2 t)\tag{5.7}$$

where  $\partial_t \alpha^p = 0$ . Note that by making a redefinition of the type

$$\begin{aligned}\lambda &= \lambda' - \frac{c_4}{c_2} f, \\ \sigma &= \sigma' + \frac{c_3}{c_2} f,\end{aligned}\tag{5.8}$$

we can without loss of generality set  $c_3 = c_4 = 0$  and drop the primes on  $\lambda$  and  $\sigma$ .

It will be convenient to split the solutions into three classes. For the first class  $c_2 \neq 0$  and  $c_3 = c_4 = 0$ , for the second  $c_2 = 0$  but  $c_3^2 + c_4^2 \neq 0$  and for the third  $c_2 = c_3 = c_4 = 0$ .

### 5.1 Solutions with $c_2 \neq 0$ , $c_3 = c_4 = 0$

For this class of solutions we have

$$\begin{aligned}\partial_t \lambda &= -c\sigma^*, \\ \partial_t \sigma &= c\lambda^*, \\ \partial_t \mu^p &= c\epsilon_{\bar{q}}^p (\mu^q)^*\end{aligned}\tag{5.9}$$

where  $c = c_2 \neq 0$ . Here we have the conditions  $\partial_t K = -cZ$  and  $\partial_t Z = cK$ .

To proceed with the analysis for these solutions, we define the 1-forms  $\phi$ ,  $\psi$  and  $L$  on the Kähler base via

$$\begin{aligned}L_p &= \frac{1}{f} (\lambda Z_p - \sigma^* K_p), & L_{\bar{p}} &= \frac{1}{f} (\lambda^* Z_{\bar{p}} - \sigma K_{\bar{p}}) \\ \psi_p &= \frac{1}{f} (\lambda^* Z_p - \sigma K_p), & \psi_{\bar{p}} &= \frac{1}{f} (\lambda Z_{\bar{p}} - \sigma^* K_{\bar{p}}), \\ \phi_p &= \frac{1}{f} (\lambda K_p + \sigma^* Z_p), & \phi_{\bar{p}} &= \frac{1}{f} (\lambda^* K_{\bar{p}} + \sigma Z_{\bar{p}}).\end{aligned}\tag{5.10}$$



The components of these 1-forms can be easily shown to be  $t$ -independent

$$\partial_t L_p = \partial_t \psi_p = \partial_t \phi_p = 0. \quad (5.11)$$

For convenience we set  $\xi^2 = |\lambda|^2 + |\sigma|^2$  and  $z = (\lambda^*)^2 + \sigma^2$ .

In order to evaluate various integrability constraints, it is useful to compute the components of the covariant derivatives:

$$\begin{aligned} \nabla_p L_q &= -\frac{1}{\sqrt{2}} \left( d\Omega_{mn} \epsilon^{mn} + \frac{3}{f^4} \chi V_I X^I \right) (K_p K_q + Z_p Z_q), \\ \nabla_p L_{\bar{q}} &= \frac{1}{\sqrt{2}} d\Omega_m{}^m (Z_p K_{\bar{q}} - K_p Z_{\bar{q}}) + \frac{c\xi^2}{\sqrt{2}f^2} \delta_{p\bar{q}} - \frac{1}{\sqrt{2}} \left( 3\frac{\chi V_I X^I}{f^4} + \frac{c}{f^6} \right) (K_p K_{\bar{q}} + Z_p Z_{\bar{q}}), \end{aligned} \quad (5.12)$$

and

$$\begin{aligned} \nabla_p \psi_q &= -\frac{1}{\sqrt{2}} \left( \frac{3\chi V_I X^I}{f^4} + \frac{c}{f^6} \right) (K_p K_q + Z_p Z_q) + \frac{1}{\sqrt{2}} d\Omega_m{}^m (Z_p K_q - K_p Z_q) \\ &\quad - \frac{\sqrt{2}ic}{f^2} \text{Im}(\lambda\sigma) \epsilon_{pq}, \end{aligned} \quad (5.13)$$

$$\nabla_p \psi_{\bar{q}} = -\frac{1}{\sqrt{2}} \left( d\Omega_{mn} \epsilon^{mn} + \frac{3}{f^4} \chi V_I X^I \right) (K_p K_{\bar{q}} + Z_p Z_{\bar{q}}) + \frac{cz^*}{\sqrt{2}f^2} \delta_{p\bar{q}}, \quad (5.14)$$

and

$$\nabla_p \phi_q = \frac{1}{\sqrt{2}} \left( d\Omega_{mn} \epsilon^{mn} + \frac{3}{f^4} \chi V_I X^I \right) (K_p Z_q - K_q Z_p) + \frac{c}{\sqrt{2}f^2} z^* \epsilon_{pq}, \quad (5.15)$$

$$\begin{aligned} \nabla_p \phi_{\bar{q}} &= -\frac{1}{\sqrt{2}} d\Omega_m{}^m (K_p K_{\bar{q}} + Z_p Z_{\bar{q}}) + \frac{1}{\sqrt{2}} \left( \frac{3\chi V_I X^I}{f^4} + \frac{c}{f^6} \right) (K_p Z_{\bar{q}} - K_{\bar{q}} Z_p) \\ &\quad + \frac{\sqrt{2}ic}{f^2} \text{Im}(\lambda\sigma) \delta_{p\bar{q}}. \end{aligned} \quad (5.16)$$

It immediately follows that  $dL = 0$  and  $\phi$  defines a Killing vector on the Kähler base space. In particular, setting  $K^2 = 2K_p K^p$ ,  $L$  is exact and satisfies

$$dK^2 = \sqrt{2}cL. \quad (5.17)$$

The first integrability condition we shall examine is obtained by considering the constraints (4.24), (4.25), (4.27) and (4.28). These are equivalent to

$$d\left(\frac{X_I}{f^2}\right) = \frac{\sqrt{2}}{K^2} \left( \chi V_I (\psi - L) - c \frac{X_I}{f^2} L \right). \quad (5.18)$$

Taking the exterior derivative of this equation, we obtain the constraint

$$d\psi = 0. \quad (5.19)$$

Hence, using (5.13) and (5.14) we obtain the constraints

$$d\Omega_m{}^m = \frac{4ic}{f^2 K^2} \text{Im}(\lambda\sigma), \quad (5.20)$$

and

$$\frac{1}{2}K^2(d\Omega_{\bar{m}\bar{n}}\epsilon^{\bar{m}\bar{n}} - d\Omega_{mn}\epsilon^{mn}) - \frac{2ic}{f^2}\text{Im}z = 0. \quad (5.21)$$

Observe also that (4.31) is equivalent to

$$d\Omega_{p\bar{q}} = \frac{2ic}{f^2K^2}\text{Im}(\lambda\sigma)\delta_{p\bar{q}} - \frac{3}{f^4K^2}\left(2\chi V_I X^I + \frac{c}{f^2}\right)(K_p Z_{\bar{q}} - Z_p K_{\bar{q}}). \quad (5.22)$$

Using (5.15) and (5.16), we compute

$$d\phi = \sqrt{2}\left(3\frac{\chi V_I X^I}{f^4} + \frac{c}{f^6}\right)K \wedge Z + \frac{1}{\sqrt{2}}\left(K^2\left(d\Omega_{mn}\epsilon^{mn} - \frac{c}{f^6}\right) - 2\frac{cz^*}{f^2}\right)J. \quad (5.23)$$

To proceed, we impose the integrability condition,  $d^2\phi = 0$ . We note the following useful identities:

$$\begin{aligned} \sqrt{2}d\left(\frac{3}{f^4}\chi V_I X^I + \frac{c}{f^6}\right) &= \frac{3V_I}{f^2K^2}\left(-3\chi^2(Q^{IJ} - 2X^I X^J)V_J + 2c\frac{\chi}{f^2}X^I\right)(\psi - L) \\ &\quad - \frac{6c}{f^4K^2}\left(2\chi V_I X^I + \frac{c}{f^2}\right)L \end{aligned} \quad (5.24)$$

and

$$\begin{aligned} L \wedge K \wedge Z &= \frac{K^2}{2}(L - \psi) \wedge J, \\ \psi \wedge K \wedge Z &= -\frac{K^2}{2}(L - \psi) \wedge J, \\ d(K \wedge Z) &= -\frac{c}{\sqrt{2}}(3\psi - L) \wedge J \end{aligned} \quad (5.25)$$

from whence we obtain

$$\epsilon^{pq}d\left(\left(\frac{3\chi V_I X^I}{f^4} + \frac{c}{f^6}\right)K \wedge Z\right)_{pq\bar{\ell}} = -d\left(K^2\left(\frac{6\chi V_I X^I}{f^4} + \frac{c}{f^6}\right)\right)_{\bar{\ell}}. \quad (5.26)$$

Using this expression, the constraint  $d^2\phi = 0$  implies that

$$d\Omega_{mn}\epsilon^{mn} = \frac{\sqrt{2}\theta}{K^2} + \frac{2cz^*}{f^2K^2} - \frac{6\chi V_I X^I}{f^4} \quad (5.27)$$

for real constant  $\theta$ .

Next we consider the integrability condition  $d^2\Omega = 0$ . It is straightforward to show that  $d\Omega$  satisfies

$$d\Omega = -\sqrt{2}d\left(\frac{\phi}{K^2}\right) + \frac{c}{f^2}\left(\frac{1}{2f^4} - \frac{\xi^2}{K^2}\right)\left(J - \frac{2}{K^2}K \wedge Z\right) + \frac{\theta}{\sqrt{2}K^2}J. \quad (5.28)$$

Hence the integrability condition  $d^2\Omega = 0$  implies that

$$d\left(\left(\frac{c}{2f^6} - \frac{c\xi^2}{f^2K^2}\right)\left(J - \frac{2}{K^2}K \wedge Z\right) + \frac{\theta}{\sqrt{2}K^2}J\right) = 0. \quad (5.29)$$

We observe that

$$\nabla_p \left( \frac{1}{f^6} - \frac{2\xi^2}{f^2 K^2} \right) = \frac{2}{(K^2)^2} \left( \theta + \frac{\sqrt{2}}{f^2} cz^* \right) \psi_p + \frac{2\sqrt{2}c}{f^2 K^2} \left( \frac{\xi^2}{K^2} - \frac{1}{f^4} \right) L_p + \frac{8\sqrt{2}ic}{f^2 (K^2)^2} \text{Im}(\lambda\sigma) \phi_p. \quad (5.30)$$

It is then straightforward but tedious to show that (5.29) implies

$$\text{Im } \lambda\sigma = 0. \quad (5.31)$$

Using this condition the above relations simplify and we obtain

$$\begin{aligned} d\Omega_m{}^m &= 0, \\ \nabla_p L_q &= - \left( \frac{\theta}{K^2} + \frac{\sqrt{2}c}{f^2 K^2} z^* - \frac{3}{\sqrt{2}f^4} \chi V_I X^I \right) (K_p K_q + Z_p Z_q), \\ \nabla_p L_{\bar{q}} &= - \frac{1}{\sqrt{2}} \left( \frac{3\chi}{f^4} V_I X^I + \frac{c}{f^6} \right) (K_p K_{\bar{q}} + Z_p Z_{\bar{q}}) + \frac{c}{\sqrt{2}f^2} \xi^2 \delta_{p\bar{q}}, \\ \nabla_p \psi_q &= - \frac{1}{\sqrt{2}} \left( \frac{3\chi}{f^4} V_I X^I + \frac{c}{f^6} \right) (K_p K_q + Z_p Z_q), \\ \nabla_p \psi_{\bar{q}} &= - \left( \frac{\theta}{K^2} + \frac{\sqrt{2}c}{f^2 K^2} z^* - \frac{3}{\sqrt{2}f^4} \chi V_I X^I \right) (K_p K_{\bar{q}} + Z_p Z_{\bar{q}}) + \frac{cz^*}{\sqrt{2}f^2} \delta_{p\bar{q}}, \\ \nabla_p \phi_q &= \left( \frac{\theta}{K^2} + \frac{\sqrt{2}c}{f^2 K^2} z^* - \frac{3}{\sqrt{2}f^4} \chi V_I X^I \right) (K_p Z_q - K_q Z_p) + \frac{cz^*}{\sqrt{2}f^2} \epsilon_{pq}, \\ \nabla_p \phi_{\bar{q}} &= \frac{1}{\sqrt{2}} \left( \frac{3\chi}{f^4} V_I X^I + \frac{c}{f^6} \right) (K_p Z_{\bar{q}} - K_{\bar{q}} Z_p). \end{aligned} \quad (5.32)$$

The components of  $d\Omega$  are therefore given by

$$\begin{aligned} d\Omega_{p\bar{q}} &= - \frac{1}{f^4 K^2} \left( 6\chi V_I X^I + \frac{3c}{f^2} \right) (K_p Z_{\bar{q}} - Z_p K_{\bar{q}}) \\ d\Omega_{mn} \epsilon^{mn} &= \frac{2c}{f^2 K^2} z^* - \frac{6\chi}{f^4} V_I X^I + \frac{\sqrt{2}\theta}{K^2}. \end{aligned} \quad (5.33)$$

Using the expressions for  $d\Omega$  which we have obtained, we next examine (4.34) and (4.35). These may be rewritten as

$$\nabla_p \left( \frac{\text{Re } \lambda}{f} \right) = - \frac{1}{2} \left( \frac{\theta}{K^2} + \frac{c}{\sqrt{2}f^6} + \frac{\sqrt{2}c}{f^2 K^2} z^* \right) Z_p - c\Omega_p \text{Re} \left( \frac{\sigma}{f} \right) \quad (5.34)$$

and

$$\nabla_p \left( \frac{\text{Re } \sigma}{f} \right) = - \frac{1}{2} \left( \frac{\theta}{K^2} + \frac{c}{\sqrt{2}f^6} + \frac{\sqrt{2}c}{f^2 K^2} z^* \right) K_p + c\Omega_p \text{Re} \left( \frac{\lambda}{f} \right). \quad (5.35)$$

We note the useful identities:

$$\nabla_p \Sigma = \frac{3c\chi}{f^4 K^2} V_I X^I (\psi_p + L_p) - \frac{3\sqrt{2}c}{K^2} \Sigma L_p \quad (5.36)$$

$$\nabla_p \Sigma^* = - \frac{c}{f^4 K^2} \left( 3\chi V_I X^I + \frac{2c}{f^2} \right) (\psi_p + L_p) - \frac{\sqrt{2}c}{K^2} \Sigma^* L_p \quad (5.37)$$

where we have set  $\Sigma = \frac{\theta}{K^2} + \frac{c}{\sqrt{2}f^6} + \frac{\sqrt{2}c}{f^2K^2}z^*$ . Then from the integrability condition  $\epsilon^{mn}\nabla_m\nabla_n(\text{Re}\frac{\lambda}{f}) = 0$ , we find the constraint

$$c\sigma^* \left( \frac{\theta}{K^2} + \frac{c}{\sqrt{2}f^6} - \frac{\sqrt{2}c}{f^2K^2}\xi^2 \right) + \left( 3\sqrt{2}\frac{c\chi V_I X^I}{f^4} - 2\frac{c\theta}{K^2} \right) \text{Re}\sigma + \frac{2ic}{fK^2}Z^p\omega_{p,mn}\epsilon^{mn} \text{Im}z = 0. \quad (5.38)$$

Note that

$$\partial_t \text{Im}z = \partial_t \xi^2 = 0,$$

then upon differentiating (5.38) with respect to  $t$  gives

$$c\lambda \left( \frac{\theta}{K^2} + \frac{c}{\sqrt{2}f^6} - \frac{\sqrt{2}c}{f^2K^2}\xi^2 \right) + \left( 3\sqrt{2}\frac{c\chi V_I X^I}{f^4} - 2\frac{c\theta}{K^2} \right) \text{Re}\lambda + \frac{2ic}{fK^2}K^p\omega_{p,mn}\epsilon^{mn} \text{Im}z = 0. \quad (5.39)$$

It turns out that the constraints (5.38) and (5.39) are also sufficient to ensure that

$$\nabla_{[p}\nabla_{q]}\left(\text{Re}\frac{\lambda}{f}\right) = \nabla_{[p}\nabla_{q]}\left(\text{Re}\frac{\sigma}{f}\right) = 0. \quad (5.40)$$

Next, note that the constraints (4.32) and (4.33) can be used to write the gauge field strengths  $F^I$  as

$$\begin{aligned} F^I &= d(f^2 X^I(dt + \Omega)) + \frac{6\chi}{f^2}V_J\left(X^I X^J - \frac{1}{2}Q^{IJ}\right)\left(\frac{1}{K^2}K \wedge Z - J\right) \\ &+ \frac{c}{f^4}X^I\left(\frac{2}{K^2}K \wedge Z - J\right). \end{aligned} \quad (5.41)$$

and note that as  $(L - \psi) \wedge \left(\frac{1}{K^2}K \wedge Z - J\right) = 0$  it follows that

$$d\left(X^I X^J - \frac{1}{2}Q^{IJ}\right)V_J \wedge \left(\frac{1}{K^2}K \wedge Z - J\right) = 0.$$

It is then straightforward to show that the Bianchi identity  $dF^I = 0$  follows automatically from the constraints we have obtained. To proceed further, it is useful to consider the cases for which  $\text{Im}z = 0$  and  $\text{Im}z \neq 0$  separately. Observe that  $\text{Im}z = 0$  implies that  $\lambda$  and  $\sigma$  are either both real or both imaginary.

### 5.1.1 Solutions with $\text{Im}z \neq 0$

In order to introduce a local co-ordinate system for solutions with  $\text{Im}z \neq 0$ , recall that  $\phi$  is a Killing vector on the Kähler base space. Furthermore, as  $\psi = i_\phi J$ , the closure of  $\psi$  implies that  $\phi$  preserves the Kähler form;

$$\mathcal{L}_\phi J = 0. \quad (5.42)$$

It is also straightforward to show that

$$\mathcal{L}_\phi X_I = \mathcal{L}_\phi f = \mathcal{L}_\phi d\Omega = \mathcal{L}_\phi F^I = 0. \quad (5.43)$$

Hence it follows that  $\phi$  defines a symmetry of the full five dimensional solution. As  $\text{Im } z \neq 0$ , (5.38) and (5.39) can be inverted to obtain

$$\omega_{p,mn}\epsilon^{mn} = \frac{i}{\sqrt{2}\text{Im } z} \left( \frac{c}{f^4} - \frac{2c\xi^2}{K^2} + 3\chi V_I X^I \left( \frac{1}{f^2} + \frac{z}{f^2\xi^2} \right) - \frac{\sqrt{2}z\theta f^2}{\xi^2 K^2} \right) \phi_p. \quad (5.44)$$

It is convenient to define the real 1-forms  $\hat{L}$  and  $\hat{\phi}$  by

$$\begin{aligned} \hat{L}_p &= iL_p, & \hat{L}_{\bar{p}} &= -iL_{\bar{p}}, \\ \hat{\phi}_p &= i\phi_p, & \hat{\phi}_{\bar{p}} &= -i\phi_{\bar{p}}. \end{aligned} \quad (5.45)$$

It is then straightforward, but tedious, to show that

$$d \left( \frac{f^2}{(K^2)^2 \text{Im } z} \hat{L} \right) = 0, \quad (5.46)$$

and also that

$$[\phi, L] = [\phi, \hat{L}] = [\phi, \hat{\phi}] = 0. \quad (5.47)$$

In addition, we find that

$$d\phi = \left( \frac{3\chi V_I X^I}{\sqrt{2}f^2\xi^2} + \frac{c}{\sqrt{2}f^4\xi^2} \right) (\phi \wedge L + \hat{\phi} \wedge \hat{L}) + \left( \theta - \frac{3}{\sqrt{2}f^4} \chi K^2 V_I X^I \right) J. \quad (5.48)$$

Hence we define the following orthonormal basis on the Kähler base space

$$\mathbf{e}^1 = \frac{\phi}{H}, \quad \mathbf{e}^2 = \frac{L}{H}, \quad \mathbf{e}^3 = \frac{\hat{\phi}}{H}, \quad \mathbf{e}^4 = \frac{\hat{L}}{H} \quad (5.49)$$

where

$$H^2 = \frac{K^2 \xi^2}{f^2}. \quad (5.50)$$

As  $\phi, \hat{\phi}$  are commuting vector fields, we can choose co-ordinates  $\tau, \eta$  such that

$$\phi = \frac{\partial}{\partial \tau}, \quad \hat{\phi} = \frac{\partial}{\partial \eta}. \quad (5.51)$$

Then, defining

$$v = \frac{K^2}{\sqrt{2}c}, \quad (5.52)$$

we have

$$L = dv \quad (5.53)$$

and from (5.46) we see that there must be a function  $u$  such that

$$\hat{L} = \frac{\text{Im } z}{\sqrt{2}f^2c} (K^2)^2 du. \quad (5.54)$$

As  $\phi, \hat{\phi}, L, \hat{L}$  are orthogonal, it follows that  $(\tau, \eta, u, v)$  form a local co-ordinate system on the base space. One can then write

$$\begin{aligned} \phi &= H^2(d\tau + \alpha_1 du + \alpha_2 dv), \\ \hat{\phi} &= H^2(d\eta + \beta_1 du + \beta_2 dv). \end{aligned} \quad (5.55)$$

As  $\phi$  is a Killing vector, the functions  $H$ ,  $f^{-2}(K^2)^2 \text{Im } z$ ,  $\alpha_1$ ,  $\alpha_2$ ,  $\beta_1$  and  $\beta_2$  do not depend on  $\tau$  (or  $t$ ). Furthermore,

$$\mathcal{L}_{\hat{\phi}} f = \mathcal{L}_{\hat{\phi}} X_I = \mathcal{L}_{\hat{\phi}} \left( \frac{(K^2)^2}{f^2} \text{Im } z \right) = 0.$$

Therefore  $f$ ,  $(K^2)^2 \text{Im } z$  and  $X_I$  are functions of  $u$  and  $v$  only. However, there is a non-trivial  $\eta$ -dependence in  $\alpha_1, \alpha_2, \beta_2, \beta_2$ ;  $\hat{\phi}$  is not a Killing vector.

It is also useful to observe that the identity  $\text{Im } \lambda \sigma = 0$  implies

$$(\text{Re } z)^2 + (\text{Im } z)^2 = \xi^4, \tag{5.56}$$

and hence we shall set

$$\begin{aligned} \cos Y &= \frac{\text{Re } z}{\xi^2}, \\ \sin Y &= \frac{\text{Im } z}{\xi^2}. \end{aligned} \tag{5.57}$$

Here  $Y$  is a real function which satisfies

$$\mathcal{L}_{\phi} Y = \mathcal{L}_{\hat{\phi}} Y = 0 \tag{5.58}$$

which implies  $Y = Y(u, v)$ . With these conventions, it is straightforward to compute

$$\begin{aligned} \frac{\partial H^2}{\partial u} &= H^2 v \sin^2 Y \left( 3 \frac{\chi c v V_I X^I}{f^4} - \theta \right), \\ \frac{\partial H^2}{\partial v} &= -\frac{c v}{f^4} \left( 3 \chi V_I X^I + \frac{c}{f^2} \right) + \cos Y \left( 3 \frac{\chi c v V_I X^I}{f^4} - \theta \right), \end{aligned} \tag{5.59}$$

and

$$\begin{aligned} \frac{\partial Y}{\partial u} &= \sin Y \left( -H^2 + 3 \frac{\chi c v^2 V_I X^I}{f^4} + \frac{c^2 v^2}{f^6} \right) + \frac{v}{2} \sin 2Y \left( 3 \frac{\chi c v V_I X^I}{f^4} - \theta \right), \\ \frac{\partial Y}{\partial v} &= -\frac{1}{H^2} \sin Y \left( 3 \frac{\chi c v V_I X^I}{f^4} - \theta \right). \end{aligned} \tag{5.60}$$

Also note that (5.18) can be rewritten as

$$\begin{aligned} \frac{\partial}{\partial u} \left( \frac{X_I}{f^2} \right) &= \frac{\chi H^2 V_I \sin^2 Y}{c}, \\ \frac{\partial}{\partial v} \left( \frac{X_I}{f^2} \right) &= \frac{1}{v} \left( \frac{\chi V_I (\cos Y - 1)}{c} - \frac{X_I}{f^2} \right). \end{aligned} \tag{5.61}$$

If  $\theta \neq 0$ , then this constraint can be integrated up to give

$$X_I = f^2 \left( \frac{q_I}{v} + \frac{\chi}{c} \left( \frac{c^2 v}{f^6 \theta} - \frac{H^2}{\theta v} - 1 \right) V_I \right). \tag{5.62}$$

for constant  $q_I$ .

To proceed, consider the equation (5.48), which can be rewritten as

$$d\phi = \frac{cv}{f^6} (3\chi f^2 V_I X^I + c) (\mathbf{e}^1 \wedge \mathbf{e}^2 + \mathbf{e}^3 \wedge \mathbf{e}^4) - \left( 3\frac{\chi cv V_I X^I}{f^4} - \theta \right) J. \quad (5.63)$$

Taking the self-dual projection of (5.63) yields the constraints

$$\begin{aligned} \frac{\partial \alpha_1}{\partial \eta} &= \frac{1}{2} v \sin 2Y \left( 3\frac{\chi cv V_I X^I}{f^4} - \theta \right) + \frac{cv^2 \sin Y}{f^6} (3\chi f^2 V_I X^I + c), \\ \frac{\partial \alpha_2}{\partial \eta} &= -\frac{\sin Y}{H^2} \left( 3\frac{\chi cv V_I X^I}{f^4} - \theta \right) \end{aligned} \quad (5.64)$$

together with

$$\frac{\partial \alpha_2}{\partial u} - \frac{\partial \alpha_1}{\partial v} + \beta_2 \frac{\partial \alpha_1}{\partial \eta} - \beta_1 \frac{\partial \alpha_2}{\partial \eta} = 0. \quad (5.65)$$

Using these constraints, the anti-self-dual projection of (5.63) fixes  $J$  to be given by

$$J = \cos Y (\mathbf{e}^1 \wedge \mathbf{e}^2 - \mathbf{e}^3 \wedge \mathbf{e}^4) + \sin Y (\mathbf{e}^1 \wedge \mathbf{e}^4 - \mathbf{e}^2 \wedge \mathbf{e}^3). \quad (5.66)$$

Imposing the covariant constancy condition  $\nabla J = 0$  imposes two additional constraints:

$$\frac{\partial \beta_2}{\partial u} - \frac{\partial \beta_1}{\partial v} + \beta_2 \frac{\partial \beta_1}{\partial \eta} - \beta_1 \frac{\partial \beta_2}{\partial \eta} = 0, \quad (5.67)$$

and

$$\sin Y \frac{\partial \beta_1}{\partial \eta} + \frac{1}{2} H^2 v \sin 2Y \frac{\partial \beta_2}{\partial \eta} = -\frac{cv^2}{2f^6} \sin 2Y (3\chi f^2 V_I X^I + c). \quad (5.68)$$

Finally, we compare the spin connection components  $\omega_{p,mn}\epsilon^{mn}$  computed in this basis with the expression given in (5.44), noting that  $\omega_{p,mn}\epsilon^{mn} = -\frac{1}{2}\omega_{p,\mu\nu}J^{\mu\nu}$ .

This implies that

$$\begin{aligned} -\cos Y \frac{\partial \beta_1}{\partial \eta} + H^2 v \sin^2 Y \frac{\partial \beta_2}{\partial \eta} &= \frac{cv^2}{f^6} \cos^2 Y (3\chi f^2 V_I X^I + c) \\ &\quad + v \cos Y \left( \frac{3\chi cv V_I X^I}{f^4} - \theta \right) - H^2. \end{aligned} \quad (5.69)$$

Then from (5.68) and (5.69) we obtain

$$\begin{aligned} \frac{\partial \beta_2}{\partial \eta} &= \frac{1}{H^2} \cos Y \left( \frac{3\chi cv V_I X^I}{f^4} - \theta \right) - \frac{1}{v}, \\ \frac{\partial \beta_1}{\partial \eta} &= -v \cos Y \left( H^2 \frac{\partial \beta_2}{\partial \eta} + \frac{cv}{f^6} (3\chi f^2 V_I X^I + c) \right). \end{aligned} \quad (5.70)$$

Then (5.67) and (5.70) can be integrated up to give

$$\beta_1 = -\eta \cot Y \frac{\partial Y}{\partial u}, \quad \beta_2 = -\eta \left( \cot Y \frac{\partial Y}{\partial v} + \frac{1}{v} \right) \quad (5.71)$$

and (5.64) and (5.65) then imply

$$\alpha_1 = \eta \left( \frac{\partial Y}{\partial u} + H^2 \sin Y \right), \quad \alpha_2 = \eta \frac{\partial Y}{\partial v}. \quad (5.72)$$

Hence, in these co-ordinates, the orthonormal basis of the Kähler base space is

$$\begin{aligned}
 \mathbf{e}^1 &= H \left( d\tau + \eta \left( \frac{\partial Y}{\partial u} + H^2 \sin Y \right) du + \eta \frac{\partial Y}{\partial v} dv \right), \\
 \mathbf{e}^2 &= \frac{1}{H} dv, \\
 \mathbf{e}^3 &= H \left( d\eta - \eta \cot Y \frac{\partial Y}{\partial u} du - \eta \left( \cot Y \frac{\partial Y}{\partial v} + \frac{1}{v} \right) dv \right) \\
 \mathbf{e}^4 &= H v \sin Y du
 \end{aligned} \tag{5.73}$$

and if  $\theta \neq 0$ , then  $J$  can be written as

$$J = d \left( \left( \frac{H^2}{\theta} - \frac{c^2 v^2}{\theta f^6} \right) d\tau + \eta \sin Y dv - \frac{1}{2} \eta H^2 v \sin 2Y du \right).$$

By considering (5.28),  $\Omega$  is fixed (up to a total derivative) by

$$\begin{aligned}
 \Omega &= -\frac{1}{2cv} \left( H^2 + \frac{c^2 v^2}{f^6} \right) d\tau + \frac{\eta}{cv} \left( \frac{1}{2} \theta \sin Y - H^2 \frac{\partial Y}{\partial v} \right) dv \\
 &\quad - \eta \left( \frac{H^2}{cv} \left( \frac{\partial Y}{\partial u} + H^2 \sin Y \right) + \frac{1}{2} \sin Y \left( \frac{\theta H^2}{c} \cos Y + \frac{c}{f^6} H^2 v - \frac{1}{cv} H^4 \right) \right) du
 \end{aligned} \tag{5.74}$$

and by considering (5.41) we find the gauge field strengths are given by

$$\begin{aligned}
 F^I &= d \left[ f^2 X^I (dt + \Omega) + \frac{cv X^I}{f^4} (d\tau + \eta H^2 \sin Y du) \right. \\
 &\quad \left. - \frac{3\chi\eta}{f^2} \sin Y \left( X^I X^J - \frac{1}{2} Q^{IJ} \right) V_J (-H^2 v (1 + \cos Y) du + dv) \right]
 \end{aligned} \tag{5.75}$$

### 5.1.2 Solutions with $\text{Re } \lambda = \text{Re } \sigma = 0$ , $\lambda \neq 0$ , $\sigma \neq 0$

If  $\lambda$  and  $\sigma$  are imaginary but non-vanishing, then from (5.10) we obtain  $\psi = -L$ , and the constraints on  $\lambda$  and  $\sigma$  as given in (5.34) and (5.35) imply that

$$\frac{\theta}{K^2} + \frac{c}{\sqrt{2} f^6} = \frac{\sqrt{2} c}{f^2 K^2} \xi^2. \tag{5.76}$$

Moreover, (5.18) can be integrated up to give

$$X_I = f^2 \left( -\frac{2\chi}{c} V_I + \frac{\rho_I}{K^2} \right) \tag{5.77}$$

for constants  $\rho_I$ . The Kähler form can be expressed by

$$J = d \left( \frac{\phi f^6}{\theta f^6 + \frac{c}{\sqrt{2}} K^2} \right). \tag{5.78}$$

It can also be demonstrated that  $\Omega$  is given in these cases by

$$\Omega = \left( -\frac{c}{\theta f^6 + \frac{c}{\sqrt{2}} K^2} \phi \right) + dg_2 \tag{5.79}$$



for some real function  $g_2$  with

$$F^I = d \left( f^2 X^I (dt + \Omega) + \frac{c f^2 X^I}{\theta f^6 + \frac{c}{\sqrt{2}} K^2} \phi \right) \quad (5.80)$$

and hence

$$\omega_{p,mn} \epsilon^{mn} = -\frac{3c\chi f^2 V_I X^I}{\theta f^6 + \frac{c}{\sqrt{2}} K^2} \phi_p + \partial_p g_1 \quad (5.81)$$

for some real function  $g_1$  where  $g_1, g_2$  satisfy

$$\partial_p \arctan \left( \frac{\lambda}{\sigma} \right) = \partial_p (g_1 + c g_2) . \quad (5.82)$$

Note that  $\arctan \left( \frac{\lambda}{\sigma} \right) = ct + H$  with  $\partial_t H = 0$ . Without loss of generality, we can work in a gauge for which  $g_2 = \frac{H}{c}$  and  $g_1 = 0$ .

It is then straightforward to prove the following identities:

$$\begin{aligned} d \left( \frac{\phi}{K^2 (c\theta + \frac{c^2}{\sqrt{2}f^6} K^2)} \right) &= -\frac{\sqrt{2}c}{(K^2)^2 (c\theta + \frac{c^2}{\sqrt{2}f^6} K^2)} \hat{L} \wedge \hat{\phi}, \\ d \left( \frac{\hat{L}}{K^2 \sqrt{c\theta + \frac{c^2}{\sqrt{2}f^6} K^2}} \right) &= \frac{\sqrt{2}c^2\theta}{(K^2)^2 (c\theta + \frac{c^2}{\sqrt{2}f^6} K^2)^{\frac{3}{2}}} \phi \wedge \hat{\phi}, \\ d \left( \frac{\hat{\phi}}{K^2 \sqrt{c\theta + \frac{c^2}{\sqrt{2}f^6} K^2}} \right) &= \frac{\sqrt{2}c^2\theta}{(K^2)^2 (c\theta + \frac{c^2}{\sqrt{2}f^6} K^2)^{\frac{3}{2}}} \hat{L} \wedge \phi. \end{aligned} \quad (5.83)$$

There are then three cases to consider.

i) If  $c\theta > 0$  then define

$$\begin{aligned} \sigma^1 &= \frac{\sqrt{2}c^2\theta}{K^2 (c\theta + \frac{c^2}{\sqrt{2}f^6} K^2)} \phi, \\ \sigma^2 &= \frac{\sqrt{2}c^3\theta}{K^2 \sqrt{c\theta + \frac{c^2}{\sqrt{2}f^6} K^2}} \hat{L}, \\ \sigma^3 &= \frac{\sqrt{2}c^3\theta}{K^2 \sqrt{c\theta + \frac{c^2}{\sqrt{2}f^6} K^2}} \hat{\phi}. \end{aligned} \quad (5.84)$$

It is then straightforward to show that

$$\begin{aligned} d\sigma^i &= -\frac{1}{2} \epsilon_{ijk} \sigma^j \wedge \sigma^k, \\ \mathcal{L}_L \sigma^i &= 0. \end{aligned} \quad (5.85)$$

ii) If  $\theta = 0$  then define

$$\begin{aligned}\sigma^1 &= -\frac{f^6}{c^3(K^2)^2}\phi, \\ \sigma^2 &= \frac{f^3}{K^2\sqrt{\frac{c^2}{\sqrt{2}}K^2}}\hat{L}, \\ \sigma^3 &= \frac{f^3}{K^2\sqrt{\frac{c^2}{\sqrt{2}}K^2}}\hat{\phi},\end{aligned}\tag{5.86}$$

where

$$\begin{aligned}d\sigma^1 &= \sigma^2 \wedge \sigma^3, \\ d\sigma^2 &= d\sigma^3 = 0, \\ \mathcal{L}_L\sigma^i &= 0.\end{aligned}\tag{5.87}$$

iii) If  $c\theta < 0$  then define

$$\begin{aligned}\sigma^1 &= \frac{\sqrt{2}c^2\theta}{K^2(c\theta + \frac{c^2}{\sqrt{2}f^6}K^2)}\phi, \\ \sigma^2 &= \frac{\sqrt{-2c^3\theta}}{K^2\sqrt{c\theta + \frac{c^2}{\sqrt{2}f^6}K^2}}\hat{L}, \\ \sigma^3 &= \frac{\sqrt{-2c^3\theta}}{K^2\sqrt{c\theta + \frac{c^2}{\sqrt{2}f^6}K^2}}\hat{\phi}.\end{aligned}\tag{5.88}$$

so that

$$\begin{aligned}d\sigma^1 &= \sigma^2 \wedge \sigma^3, \\ d\sigma^2 &= \sigma^1 \wedge \sigma^3, \\ d\sigma^3 &= -\sigma^1 \wedge \sigma^2, \\ \mathcal{L}_L\sigma^i &= 0.\end{aligned}\tag{5.89}$$

Hence the 3-manifold with metric  $\frac{1}{4}((\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2)$  is either  $\mathbb{S}^3$ , the Nil-manifold or  $\mathbb{H}^3$  according as to whether  $c\theta > 0$ ,  $\theta = 0$  or  $c\theta < 0$  respectively.

### 5.1.3 Solutions with $\text{Im } \lambda = \text{Im } \sigma = 0$ , $\lambda \neq 0$ , $\sigma \neq 0$

If  $\lambda$  and  $\sigma$  are real but non-vanishing, then  $\psi = L$ , and from (5.34) and (5.35) we obtain

$$V_I X^I = \frac{1}{3\sqrt{2}\chi} \left( \frac{f^4\theta}{K^2} + \frac{\sqrt{2}cf^2}{K^2}\xi^2 - \frac{c}{\sqrt{2}f^2} \right).\tag{5.90}$$

Moreover, from (5.18) we obtain

$$\begin{aligned}d\left(\frac{X_I}{f^2}\right) &= -\frac{\sqrt{2}}{f^2K^2}cX_I L, \\ d\left(\frac{1}{f^2}\right) &= -\frac{dK^2}{f^2K^2}.\end{aligned}\tag{5.91}$$

This implies that  $d(K^2 f^{-2}) = 0$ , and hence without loss of generality we can set  $K^2 = f^2$  and the scalars are therefore constants. Furthermore, we also find that

$$J = d\left(-\frac{f^2 \phi}{\sqrt{2} c \xi^2}\right), \quad (5.92)$$

and

$$d\Omega = -d\left(\left(\frac{1}{\sqrt{2} f^2} + \frac{1}{2\sqrt{2} f^4 \xi^2} + \frac{\theta}{2c\xi^2}\right) \phi\right) \quad (5.93)$$

with

$$F^I = d\left(f^2 X^I (dt + \Omega) + \frac{3\sqrt{2}\chi}{c\xi^2} \left(X^I X^J - \frac{1}{2} Q^{IJ}\right) V_J \phi + \frac{1}{\sqrt{2} f^2 \xi^2} X^I \phi\right), \quad (5.94)$$

and hence we can work in a gauge for which

$$\omega_{p,mn} \epsilon^{mn} = -\frac{3\chi}{\sqrt{2}\xi^2} \left(\frac{6\chi}{c} \left(X^I X^J - \frac{1}{2} Q^{IJ}\right) V_I V_J + \frac{1}{f^2} V_I X^I\right) \phi_p. \quad (5.95)$$

It is then straightforward to prove the following identities:

$$\begin{aligned} d\left(\frac{\phi}{\xi^2}\right) &= -\frac{\sqrt{2}c}{f^2 \xi^2} \hat{L} \wedge \hat{\phi}, \\ d\left(\frac{\hat{L}}{f\xi}\right) &= -\frac{1}{f\xi^3} \left(\theta + \frac{9\sqrt{2}\chi^2}{c} \left(X^I X^J - \frac{1}{2} Q^{IJ}\right) V_I V_J\right) \phi \wedge \hat{\phi}, \\ d\left(\frac{\hat{\phi}}{f\xi}\right) &= -\frac{1}{f\xi^3} \left(\theta + \frac{9\sqrt{2}\chi^2}{c} \left(X^I X^J - \frac{1}{2} Q^{IJ}\right) V_I V_J\right) \hat{L} \wedge \phi. \end{aligned} \quad (5.96)$$

There are then three cases to consider.

- i) If  $c\theta + 9\sqrt{2}\chi^2(X^I X^J - \frac{1}{2}Q^{IJ})V_I V_J < 0$  then define

$$\begin{aligned} \sigma^1 &= -\frac{\theta + \frac{9\sqrt{2}\chi^2}{c}(X^I X^J - \frac{1}{2}Q^{IJ})V_I V_J}{\xi^2} \phi, \\ \sigma^2 &= \frac{\sqrt{-\sqrt{2}(c\theta + 9\sqrt{2}\chi^2(X^I X^J - \frac{1}{2}Q^{IJ})V_I V_J)}}{f\xi} \hat{L}, \\ \sigma^3 &= \frac{\sqrt{-\sqrt{2}(c\theta + 9\sqrt{2}\chi^2(X^I X^J - \frac{1}{2}Q^{IJ})V_I V_J)}}{f\xi} \hat{\phi}. \end{aligned} \quad (5.97)$$

It is then straightforward to show that

$$\begin{aligned} d\sigma^i &= -\frac{1}{2} \epsilon_{ijk} \sigma^j \wedge \sigma^k, \\ \mathcal{L}_L \sigma^i &= 0. \end{aligned} \quad (5.98)$$

ii) If  $c\theta + 9\sqrt{2}\chi^2(X^I X^J - \frac{1}{2}Q^{IJ})V_I V_J = 0$  then define

$$\begin{aligned}\sigma^1 &= -\frac{\phi}{\sqrt{2c\xi^2}}, \\ \sigma^2 &= \frac{\hat{L}}{f\xi}, \\ \sigma^3 &= \frac{\hat{\phi}}{f\xi},\end{aligned}\tag{5.99}$$

where

$$\begin{aligned}d\sigma^1 &= \sigma^2 \wedge \sigma^3, & d\sigma^2 &= d\sigma^3 = 0, \\ \mathcal{L}_L \sigma^i &= 0.\end{aligned}\tag{5.100}$$

iii) If  $c\theta + 9\sqrt{2}\chi^2(X^I X^J - \frac{1}{2}Q^{IJ})V_I V_J > 0$  then define

$$\begin{aligned}\sigma^1 &= -\frac{\theta + \frac{9\sqrt{2}\chi^2}{c}(X^I X^J - \frac{1}{2}Q^{IJ})V_I V_J}{\xi^2}\phi, \\ \sigma^2 &= \frac{\sqrt{\sqrt{2}(c\theta + 9\sqrt{2}\chi^2(X^I X^J - \frac{1}{2}Q^{IJ})V_I V_J)}}{f\xi}\hat{L}, \\ \sigma^3 &= \frac{\sqrt{\sqrt{2}(c\theta + 9\sqrt{2}\chi^2(X^I X^J - \frac{1}{2}Q^{IJ})V_I V_J)}}{f\xi}\hat{\phi},\end{aligned}\tag{5.101}$$

so that

$$\begin{aligned}d\sigma^1 &= \sigma^2 \wedge \sigma^3, \\ d\sigma^2 &= \sigma^1 \wedge \sigma^3, \\ d\sigma^3 &= -\sigma^1 \wedge \sigma^2, \\ \mathcal{L}_L \sigma^i &= 0.\end{aligned}\tag{5.102}$$

Hence the 3-manifold with metric  $\frac{1}{4}((\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2)$  is either  $\mathbb{S}^3$ , the Nil-manifold or  $\mathbb{H}^3$  according as to whether  $c\theta + 9\sqrt{2}\chi^2(X^I X^J - \frac{1}{2}Q^{IJ})V_I V_J < 0$ ,  $c\theta + 9\sqrt{2}\chi^2(X^I X^J - \frac{1}{2}Q^{IJ})V_I V_J = 0$  or  $c\theta + 9\sqrt{2}\chi^2(X^I X^J - \frac{1}{2}Q^{IJ})V_I V_J > 0$  respectively.

#### 5.1.4 Solutions with $\lambda = \sigma = 0$

If  $\lambda = \sigma = 0$ , then  $f$  and the scalars  $X^I$  are constant and  $K^2$  is constant. Without loss of generality, we set  $f = K^2 = 1$ . The following constraints also hold:

$$\theta = -\frac{c}{\sqrt{2}}, \quad 3\chi V_I X^I = -c\tag{5.103}$$

so that  $\theta \neq 0, V_I X^I \neq 0$  for these solutions. Furthermore, we also find

$$d\Omega = -cK \wedge Z,\tag{5.104}$$

and the gauge field strengths are given by

$$F^I = d(f^2 X^I(dt + \Omega)) + 6\chi V_J \left( X^I X^J - \frac{1}{2} Q^{IJ} \right) (K \wedge Z - J) + cX^I(2K \wedge Z - J) \quad (5.105)$$

Note also that  $K$  and  $Z$  satisfy

$$\begin{aligned} \nabla_{\tilde{\mu}} K_{\tilde{\nu}} &= -c\Omega_{\tilde{\mu}} Z_{\tilde{\nu}}, \\ \nabla_{\tilde{\mu}} Z_{\tilde{\nu}} &= c\Omega_{\tilde{\mu}} K_{\tilde{\nu}} \end{aligned} \quad (5.106)$$

where here  $\nabla$  denotes the covariant derivative restricted to the base space, and  $\tilde{\mu}, \tilde{\nu}$  are base space indices.

It is convenient to define

$$\begin{aligned} \hat{K}_p &= iK_p, & \hat{K}_{\bar{p}} &= -iK_{\bar{p}} \\ \hat{Z}_p &= iZ_p, & \hat{Z}_{\bar{p}} &= -iZ_{\bar{p}}. \end{aligned} \quad (5.107)$$

Then

$$J = K \wedge Z - \hat{K} \wedge \hat{Z}, \quad (5.108)$$

and

$$\nabla_{\tilde{\mu}} \hat{K}_{\tilde{\nu}} = -c\hat{\Omega}_{\tilde{\mu}} \hat{Z}_{\tilde{\nu}}, \quad \nabla_{\tilde{\mu}} \hat{Z}_{\tilde{\nu}} = c\hat{\Omega}_{\tilde{\mu}} \hat{K}_{\tilde{\nu}} \quad (5.109)$$

where  $\hat{\Omega}$  is defined by

$$\hat{\Omega}_p = \Omega_p + \frac{1}{c} \omega_{p,mn} \epsilon^{mn}, \quad \hat{\Omega}_{\bar{p}} = (\hat{\Omega}_p)^*. \quad (5.110)$$

Note that (5.109) implies that

$$d\hat{\Omega} \wedge \hat{K} = d\hat{\Omega} \wedge \hat{Z} = 0, \quad (5.111)$$

and hence

$$d\hat{\Omega} = \Psi \hat{K} \wedge \hat{Z} \quad (5.112)$$

where  $\Psi$  is fixed by comparing the integrability condition associated with (3.59) with the expression for the gauge field strengths (5.105). We find

$$\Psi = \frac{1}{c} (9\chi^2 Q^{IJ} V_I V_J - c^2). \quad (5.113)$$

Next we define

$$\begin{aligned} A_p &= \cos ct K_p + \sin ct Z_p, \\ A_{\bar{p}} &= \cos ct K_{\bar{p}} + \sin ct Z_{\bar{p}}, \\ B_p &= \epsilon_{pq} A^q = \cos ct Z_p - \sin ct K_p, \\ B_{\bar{p}} &= \cos ct Z_{\bar{p}} - \sin ct K_{\bar{p}}, \end{aligned} \quad (5.114)$$

so that

$$\partial_t A = \partial_t B = 0. \quad (5.115)$$

We also define

$$\begin{aligned}\hat{A}_p &= iA_p, & \hat{A}_{\bar{p}} &= -iA_{\bar{p}}, \\ \hat{B}_p &= iB_p, & \hat{B}_{\bar{p}} &= -iB_{\bar{p}}.\end{aligned}\tag{5.116}$$

Then  $A, B, \hat{A}, \hat{B}$  form an orthonormal time-independent basis for the Kähler base space, such that

$$\begin{aligned}\nabla_{\bar{\mu}}A_{\bar{\nu}} &= -c\Omega_{\bar{\mu}}B_{\bar{\nu}}, \\ \nabla_{\bar{\mu}}B_{\bar{\nu}} &= c\Omega_{\bar{\mu}}A_{\bar{\nu}}, \\ d\Omega &= -cA \wedge B\end{aligned}\tag{5.117}$$

and

$$\begin{aligned}\nabla_{\bar{\mu}}\hat{A}_{\bar{\nu}} &= -c\hat{\Omega}_{\bar{\mu}}\hat{B}_{\bar{\nu}}, \\ \nabla_{\bar{\mu}}\hat{B}_{\bar{\nu}} &= c\hat{\Omega}_{\bar{\mu}}\hat{A}_{\bar{\nu}}, \\ d\hat{\Omega} &= \frac{1}{c}(9\chi^2 Q^{IJ}V_I V_J - c^2)\hat{A} \wedge \hat{B}.\end{aligned}\tag{5.118}$$

Now note that

$$[A, B] = (ci_A\Omega)A + (ci_B\Omega)B.\tag{5.119}$$

It therefore follows that there exist functions  $\alpha_1, \alpha_2, \beta_1, \beta_2$  and co-ordinates  $w_1, w_2$  such that

$$A = \alpha_1 \frac{\partial}{\partial w_1} + \alpha_2 \frac{\partial}{\partial w_2}, \quad B = \beta_1 \frac{\partial}{\partial w_1} + \beta_2 \frac{\partial}{\partial w_2}.\tag{5.120}$$

Suppose that the remaining co-ordinates on the base are  $y^1, y^2$ . As  $\hat{A}$  and  $\hat{B}$  are orthogonal to  $A, B$ , there exist functions  $\rho_i, \nu_i$  such that

$$\hat{A} = \rho_i dy^i, \quad \hat{B} = \nu_i dy^i\tag{5.121}$$

for  $i = 1, 2$ .

Similarly, as

$$[\hat{A}, \hat{B}] = (ci_{\hat{A}}\hat{\Omega})\hat{A} + (ci_{\hat{B}}\hat{\Omega})\hat{B}\tag{5.122}$$

it also follows that there exist functions  $\hat{\rho}_i, \hat{\nu}_i$  and  $x^i$  for  $i = 1, 2$  such that

$$A = \hat{\rho}_i dx^i, \quad B = \hat{\nu}_i dx^i\tag{5.123}$$

for  $i = 1, 2$ . As  $A, B, \hat{A}, \hat{B}$  form an orthonormal basis for the base space, we can without loss of generality take  $x^1, x^2, y^1, y^2$  to be co-ordinates on the base space.

In principle, the functions  $\rho_i, \hat{\rho}_i, \nu_i, \hat{\nu}_i$  can depend on all the co-ordinates. However, note that

$$J = A \wedge B - \hat{A} \wedge \hat{B} = (\hat{\rho}_1 \hat{\nu}_2 - \hat{\nu}_1 \hat{\rho}_2) dx^1 \wedge dx^2 - (\rho_1 \nu_2 - \nu_1 \rho_2) dy^1 \wedge dy^2.\tag{5.124}$$

Imposing the constraint  $dJ = 0$  thus gives the conditions

$$\frac{\partial}{\partial y^i}(\hat{\rho}_1 \hat{\nu}_2 - \hat{\nu}_1 \hat{\rho}_2) = \frac{\partial}{\partial x^i}(\rho_1 \nu_2 - \nu_1 \rho_2) = 0. \quad (5.125)$$

One therefore can set

$$\Omega = d\Phi + \Omega_T \quad (5.126)$$

with  $\Omega_T = \Omega_{T_i}(x^1, x^2)dx^i$  satisfying

$$d\Omega_T = -c(\hat{\rho}_1 \hat{\nu}_2 - \hat{\nu}_1 \hat{\rho}_2)dx^1 \wedge dx^2. \quad (5.127)$$

We also set

$$\hat{\Omega} = d\hat{\Phi} + \hat{\Omega}_T \quad (5.128)$$

where  $\hat{\Omega}_T = \hat{\Omega}_{T_i}(y^1, y^2)dy^i$  satisfies

$$d\hat{\Omega}_T = \frac{1}{c}(9\chi^2 Q^{IJ} V_I V_J - c^2)(\rho_1 \nu_2 - \nu_1 \rho_2)dy^1 \wedge dy^2. \quad (5.129)$$

Here  $\Phi$  and  $\hat{\Phi}$  are functions of  $x^i, y^i$ . Next we define

$$\begin{aligned} A' &= \cos c\Phi A + \sin c\Phi B, \\ B' &= -\sin c\Phi A + \cos c\Phi B, \\ \hat{A}' &= \cos c\hat{\Phi} \hat{A} + \sin c\hat{\Phi} \hat{B}, \\ \hat{B}' &= -\sin c\hat{\Phi} \hat{A} + \cos c\hat{\Phi} \hat{B}. \end{aligned} \quad (5.130)$$

Note that  $A', B', \hat{A}', \hat{B}'$  are an orthonormal basis of the Kähler base with the property that

$$\nabla_{\bar{\mu}} A'_{\bar{\nu}} = -c\Omega_{T\bar{\mu}} B'_{\bar{\nu}}, \quad \nabla_{\bar{\mu}} B'_{\bar{\nu}} = c\Omega_{T\bar{\mu}} A'_{\bar{\nu}} \quad (5.131)$$

and

$$\nabla_{\bar{\mu}} \hat{A}'_{\bar{\nu}} = -c\hat{\Omega}_{T\bar{\mu}} \hat{B}'_{\bar{\nu}}, \quad \nabla_{\bar{\mu}} \hat{B}'_{\bar{\nu}} = c\hat{\Omega}_{T\bar{\mu}} \hat{A}'_{\bar{\nu}} \quad (5.132)$$

with

$$d\Omega_T = -cA' \wedge B', \quad d\hat{\Omega}_T = \frac{1}{c}(9\chi^2 Q^{IJ} V_I V_J - c^2)\hat{A}' \wedge \hat{B}'. \quad (5.133)$$

These constraints therefore imply that

$$\mathcal{L}_{\frac{\partial}{\partial y^i}} A' = \mathcal{L}_{\frac{\partial}{\partial y^i}} B' = \mathcal{L}_{\frac{\partial}{\partial x^i}} \hat{A}' = \mathcal{L}_{\frac{\partial}{\partial x^i}} \hat{B}' = 0. \quad (5.134)$$

Hence the Kähler base is a product of two 2-manifolds  $M_1, M_2$ , with metric

$$ds_{\mathcal{B}}^2 = ds^2(M_1) + ds^2(M_2). \quad (5.135)$$

Taking the orthonormal basis  $\mathbf{e}^1 = A', \mathbf{e}^2 = B', \mathbf{e}^3 = \hat{A}', \mathbf{e}^4 = \hat{B}'$ , the metrics on  $M_1$  and  $M_2$  are

$$ds^2(M_1) = (\mathbf{e}^1)^2 + (\mathbf{e}^2)^2, \quad ds^2(M_2) = (\mathbf{e}^3)^2 + (\mathbf{e}^4)^2. \quad (5.136)$$

It is then straightforward to compute the curvature in this basis. We find that the only non-vanishing components are fixed by

$$R_{1212} = -c^2, \quad R_{3434} = 9\chi^2 Q^{IJ} V_I V_J - c^2. \quad (5.137)$$

Therefore we conclude that  $M_1$  is  $\mathbb{H}^2$ , and  $M_2$  is  $\mathbb{H}^2, \mathbb{R}^2$  or  $\mathbb{S}^2$  depending on whether  $9\chi^2 Q^{IJ} V_I V_J - c^2 < 0, 9\chi^2 Q^{IJ} V_I V_J - c^2 = 0$  or  $9\chi^2 Q^{IJ} V_I V_J - c^2 > 0$  respectively.

**5.2 Solutions with  $c_2 = 0$  and  $c_3^2 + c_4^2 = \varrho^2 \neq 0$**

In the next case, we shall assume that  $c_3$  and  $c_4$  do not both vanish, and define the vector fields  $W, Y$  on the Kähler base via

$$\begin{aligned} W &= c_4 K - c_3 Z, \\ Y &= c_3 K + c_4 Z. \end{aligned} \tag{5.138}$$

Then note that  $W \neq 0$  and  $Y \neq 0$ , and (4.19), (4.20) and (4.22) can be rewritten in terms of  $Y$  and  $W$  as

$$\begin{aligned} \nabla_p W_{\bar{q}} &= 0, \\ \nabla_p W_q &= \frac{\varrho^2}{\sqrt{2}} \epsilon_{pq}, \\ \nabla_p Y_{\bar{q}} &= \frac{\varrho^2}{\sqrt{2}} \delta_{p\bar{q}}, \\ \nabla_p Y_q &= 0. \end{aligned} \tag{5.139}$$

Therefore we find that  $W$  is a holomorphic Killing vector on the base and satisfies

$$dW = -\sqrt{2}\varrho^2 J. \tag{5.140}$$

Moreover,  $W$  preserves the complex structure

$$\mathcal{L}_W J = 0. \tag{5.141}$$

In contrast,  $Y$  defines a closed 1-form on the base, which is conformally Killing with

$$\mathcal{L}_Y h = \sqrt{2}\varrho^2 h, \quad \mathcal{L}_Y J = \sqrt{2}\varrho^2 J. \tag{5.142}$$

Here  $h$  denotes the metric of the Kähler base. From (4.23) it is clear that  $W$  and  $Y$  commute, so that locally one can choose co-ordinates  $\phi, \psi$  so that

$$W = \frac{\partial}{\partial \phi}, \quad Y = \frac{\partial}{\partial \psi} \tag{5.143}$$

and let the remaining two co-ordinates of the base space be  $x^1, x^2$ .

Note that  $Y$  and  $W$  satisfy

$$h(Y, W) = 0, \quad Y^2 = W^2. \tag{5.144}$$

Then, one can write the metric on the Kähler base locally as

$$ds^2 = e^{\sqrt{2}\varrho^2 \psi} [S^2(d\phi + \chi_1)^2 + S^2(d\psi + \chi_2)^2 + T^2((dx^1)^2 + (dx^2)^2)] \tag{5.145}$$

where  $S = S(x^1, x^2)$ ,  $T = T(x^1, x^2)$  and  $\chi_i = \chi_{ij}(x^1, x^2) dx^j$ . By making a co-ordinate transformation of the form

$$\begin{aligned} \psi &= \psi' - \frac{1}{\sqrt{2}\varrho^2} \log S^2, \\ \phi &= \phi', \\ x^1 &= (x^1)', \\ x^2 &= (x^2)', \end{aligned} \tag{5.146}$$



one can without loss of generality set  $S = 1$  in (5.145) and drop primes throughout.

Then it is straightforward to show that the condition  $dY = 0$  implies that  $\chi_2 = 0$ , so that

$$ds^2 = e^{\sqrt{2}\varrho^2\psi} [(d\phi + \beta)^2 + d\psi^2 + T^2((dx^1)^2 + (dx^2)^2)] \quad (5.147)$$

where  $\beta = \beta_i(x^1, x^2)dx^i$ , and

$$J = -\frac{1}{\sqrt{2}\varrho^2} d\left(e^{\sqrt{2}\varrho^2\psi}(d\phi + \beta)\right). \quad (5.148)$$

The necessary and sufficient condition in order for  $J$  to be a covariantly constant complex structure is

$$T^2 = \frac{1}{\sqrt{2}\varrho^2} \left| \left( \frac{\partial\beta_2}{\partial x^1} - \frac{\partial\beta_1}{\partial x^2} \right) \right| \quad (5.149)$$

In fact, we can take  $\frac{\partial\beta_2}{\partial x^1} - \frac{\partial\beta_1}{\partial x^2} > 0$  without loss of generality, (this can be obtained, if necessary, by making the re-definition  $\beta_2 \rightarrow -\beta_2$  and  $x^2 \rightarrow -x^2$ ). So

$$T^2 = \frac{1}{\sqrt{2}\varrho^2} \left( \frac{\partial\beta_2}{\partial x^1} - \frac{\partial\beta_1}{\partial x^2} \right). \quad (5.150)$$

In addition, (4.27) and (4.28) can be rewritten as

$$\begin{aligned} \frac{1}{f}c_3 - \frac{2\sqrt{2}}{f^2}K^p\nabla_p f + 2i\chi V_I X^I (\text{Im } \sigma) &= 0, \\ \frac{1}{f}c_4 - \frac{2\sqrt{2}}{f^2}Z^{\bar{p}}\nabla_{\bar{p}} f - 2i\chi V_I X^I (\text{Im } \lambda) &= 0. \end{aligned} \quad (5.151)$$

The real portions of (5.151) imply that

$$\mathcal{L}_K f = \frac{c_3}{\sqrt{2}}f, \quad \mathcal{L}_Z f = \frac{c_4}{\sqrt{2}}f \quad (5.152)$$

and hence

$$\mathcal{L}_W f = 0, \quad \mathcal{L}_Y f = \frac{1}{\sqrt{2}}\varrho^2 f. \quad (5.153)$$

Therefore

$$f = e^{\frac{1}{\sqrt{2}}\varrho^2\psi} u(x^1, x^2) \quad (5.154)$$

for some function  $u(x^1, x^2)$ . It should be noted that although the Kähler metric  $h$  has a conformal dependence on  $\psi$ , the portion of the metric  $f^{-2}h$  which appears in the five dimensional metric does not depend on either  $\phi$  or  $\psi$ .

In order to examine the behaviour of the scalars, note that (4.26) implies that

$$X^I = X^I(x^1, x^2). \quad (5.155)$$

To proceed further, we introduce the following holomorphic basis for the Kähler base space

$$\begin{aligned} \mathbf{e}^1 &= \frac{e^{\frac{1}{\sqrt{2}}\varrho^2\psi}}{\sqrt{2}\varrho} (c_3 d\psi + c_4(d\phi + \beta) - i\varrho T dx^1), \\ \mathbf{e}^2 &= \frac{e^{\frac{1}{\sqrt{2}}\varrho^2\psi}}{\sqrt{2}\varrho} (-c_4 d\psi + c_3(d\phi + \beta) + i\varrho T dx^2) \end{aligned} \quad (5.156)$$

with  $\mathbf{e}^{\bar{1}}$ ,  $\mathbf{e}^{\bar{2}}$  obtained by complex conjugation. It is straightforward to show that in this basis

$$J = - \left( \mathbf{e}^1 \wedge \mathbf{e}^2 + \mathbf{e}^{\bar{1}} \wedge \mathbf{e}^{\bar{2}} \right) \quad (5.157)$$

as expected, and

$$\begin{aligned} K^1 &= K^{\bar{1}} = \frac{1}{\sqrt{2}\varrho} e^{\frac{1}{\sqrt{2}}\varrho^2\psi}, \\ K^2 &= K^{\bar{2}} = 0, \\ Z^1 &= Z^{\bar{1}} = 0, \\ Z^2 &= Z^{\bar{2}} = -\frac{1}{\sqrt{2}\varrho} e^{\frac{1}{\sqrt{2}}\varrho^2\psi}. \end{aligned} \quad (5.158)$$

Using this basis, one can compute explicitly the following components of the spin connection:

$$\begin{aligned} \omega_{1,mn}\epsilon^{mn} &= e^{-\frac{1}{\sqrt{2}}\varrho^2\psi} \left( -\varrho c_4 + \frac{i}{\sqrt{2}T^2} \frac{\partial T}{\partial x^2} \right), \\ \omega_{2,mn}\epsilon^{mn} &= e^{-\frac{1}{\sqrt{2}}\varrho^2\psi} \left( -\varrho c_3 + \frac{i}{\sqrt{2}T^2} \frac{\partial T}{\partial x^1} \right). \end{aligned} \quad (5.159)$$

Moreover, it is straightforward to show that the imaginary portion of (5.151) implies that

$$\sqrt{2}\chi V_I X^I \operatorname{Im} \sigma = \frac{1}{\varrho T u^2} e^{-\frac{1}{\sqrt{2}}\varrho^2\psi} \frac{\partial u}{\partial x^1} \quad (5.160)$$

and

$$\sqrt{2}\chi V_I X^I \operatorname{Im} \lambda = \frac{1}{\varrho T u^2} e^{-\frac{1}{\sqrt{2}}\varrho^2\psi} \frac{\partial u}{\partial x^2}. \quad (5.161)$$

The imaginary parts of (4.24) and (4.25) can then be rewritten as

$$\begin{aligned} \frac{\partial}{\partial x^1} \left( \frac{X_I}{u^2} \right) &= -2\sqrt{2}\chi \frac{\varrho}{u} e^{\frac{1}{\sqrt{2}}\varrho^2\psi} T V_I \operatorname{Im} \sigma, \\ \frac{\partial}{\partial x^2} \left( \frac{X_I}{u^2} \right) &= -2\sqrt{2}\chi \frac{\varrho}{u} e^{\frac{1}{\sqrt{2}}\varrho^2\psi} T V_I \operatorname{Im} \lambda. \end{aligned} \quad (5.162)$$

Note that as the  $V_I$  do not all vanish in the gauged theory, it follows that the imaginary parts of  $\lambda$  and  $\sigma$  do not depend on  $\phi$ , and depend on  $\psi$  via the factor  $e^{-\frac{1}{\sqrt{2}}\varrho^2\psi}$ . It is therefore convenient to define

$$\begin{aligned} \mathcal{G}(x^1, x^2) &= \frac{2i}{u} e^{\frac{1}{\sqrt{2}}\varrho^2\psi} \operatorname{Im} \lambda \\ \mathcal{H}(x^1, x^2) &= \frac{2i}{u} e^{\frac{1}{\sqrt{2}}\varrho^2\psi} \operatorname{Im} \sigma, \end{aligned} \quad (5.163)$$

and rewrite (5.160), (5.161) and (5.162) as

$$\chi V_I X^I \mathcal{H} = \frac{\sqrt{2}i}{\varrho T u^3} \frac{\partial u}{\partial x^1}, \quad (5.164)$$

$$\chi V_I X^I \mathcal{G} = \frac{\sqrt{2}i}{\varrho T u^3} \frac{\partial u}{\partial x^2}, \quad (5.165)$$

and

$$\begin{aligned}\frac{\partial}{\partial x^1} \left( \frac{X_I}{u^2} \right) &= \sqrt{2}i\chi\varrho T\mathcal{H}V_I, \\ \frac{\partial}{\partial x^2} \left( \frac{X_I}{u^2} \right) &= \sqrt{2}i\chi\varrho T\mathcal{G}V_I.\end{aligned}\tag{5.166}$$

We next consider the constraints (4.31), (4.36) and (4.37). These are equivalent to

$$\frac{\partial}{\partial x^2}(T\mathcal{H}) = \frac{\partial}{\partial x^1}(T\mathcal{G}),\tag{5.167}$$

and

$$\frac{\partial}{\partial x^1} \left( \frac{\mathcal{H}}{T} \right) = \frac{\partial}{\partial x^2} \left( \frac{\mathcal{G}}{T} \right), \quad \frac{\partial}{\partial x^1} \left( \frac{\mathcal{G}}{T} \right) = -\frac{\partial}{\partial x^2} \left( \frac{\mathcal{H}}{T} \right).\tag{5.168}$$

Note that (5.167) implies the integrability condition associated with (5.166), and (5.168) implies that  $T^{-1}\mathcal{H}$  and  $T^{-1}\mathcal{G}$  satisfy the Cauchy-Riemann equations. Hence,  $T^{-1}(\mathcal{H} + i\mathcal{G})$  is a holomorphic function of  $x^1 + ix^2$ .

The components of  $d\Omega$  are also fixed by (4.31), (4.36) and (4.37) to be

$$\begin{aligned}d\Omega_{1\bar{2}} &= 3u^{-4}e^{-2\sqrt{2}\varrho^2\psi}\chi V_I X^I, \\ d\Omega_{1\bar{1}} &= d\Omega_{2\bar{2}} = \varrho^2 e^{-2\sqrt{2}\varrho^2\psi}(c_4\mathcal{H} - c_3\mathcal{G}), \\ d\Omega_{12} &= e^{-2\sqrt{2}\varrho^2\psi} \left[ \varrho^2(c_4\mathcal{G} + c_3\mathcal{H}) - \frac{i}{\sqrt{2}}\varrho \left( \frac{1}{T} \frac{\partial\mathcal{G}}{\partial x^2} + \frac{1}{T^2} \mathcal{H} \frac{\partial T}{\partial x^1} \right) \right]\end{aligned}\tag{5.169}$$

with the remaining components determined by complex conjugation; this exhausts the content of (4.31), (4.36) and (4.37). Using these components, it is straightforward to compute  $d\Omega$  in the co-ordinate basis; we find

$$\begin{aligned}d\Omega &= e^{-\sqrt{2}\varrho^2\psi} \left[ \left( -\frac{i}{\sqrt{2}}\varrho \left( \frac{1}{T} \frac{\partial\mathcal{G}}{\partial x^2} + \frac{\mathcal{H}}{T^2} \frac{\partial T}{\partial x^1} \right) + 3\frac{\chi}{u^4} V_I X^I \right) d\psi \wedge (d\phi + \beta) \right. \\ &\quad + \left( -\frac{i}{\sqrt{2}}\varrho \left( T \frac{\partial\mathcal{G}}{\partial x^2} + \mathcal{H} \frac{\partial T}{\partial x^1} \right) - 3\chi \frac{T^2}{u^4} V_I X^I \right) dx^1 \wedge dx^2 \\ &\quad \left. + iT\varrho^3 (d\psi \wedge (-\mathcal{G}dx^1 + \mathcal{H}dx^2) + (d\phi + \beta) \wedge (\mathcal{H}dx^1 + \mathcal{G}dx^2)) \right].\end{aligned}\tag{5.170}$$

Using (4.32) and (4.33), the gauge field strengths for these solutions can be written as

$$F^I = d(f^2 X^I(dt + \Omega)) + 6\chi \left( X^I X^J - \frac{1}{2} Q^{IJ} \right) V_J \frac{T^2}{u^2} dx^1 \wedge dx^2\tag{5.171}$$

which satisfy  $dF^I = 0$  automatically.

Next consider the integrability condition associated with (3.59): this can be written as

$$3\chi V_I (F^I - d(f^2 X^I(dt + \Omega))) = -d(\omega_{p,mn}\epsilon^{mn}\mathbf{e}^p + \omega_{\bar{p},\bar{m}\bar{n}}\epsilon^{\bar{m}\bar{n}}\mathbf{e}^{\bar{p}}).\tag{5.172}$$

This can be evaluated to give the constraint

$$\square \log T + 2\varrho^4 T^2 = 18\chi^2 \left( X^I X^J - \frac{1}{2} Q^{IJ} \right) V_I V_J \frac{T^2}{u^2}\tag{5.173}$$

where  $\square = \left(\frac{\partial}{\partial x^1}\right)^2 + \left(\frac{\partial}{\partial x^2}\right)^2$  is the Laplacian on  $\mathbb{R}^2$ . In fact, this constraint enables (5.170) to be solved for  $\Omega$  (up to a total derivative); we find

$$\Omega = -\frac{e^{-\sqrt{2}\varrho^2\psi}}{\sqrt{2}} \left[ iT\varrho(-\mathcal{G}dx^1 + \mathcal{H}dx^2) + \left(-\frac{i}{\sqrt{2}}\frac{1}{\varrho} \left(\frac{1}{T}\frac{\partial\mathcal{G}}{\partial x^2} + \frac{\mathcal{H}}{T^2}\frac{\partial T}{\partial x^1}\right) + \frac{3\chi}{\varrho^2 u^2} V_I X^I\right) (d\phi + \beta) \right] \quad (5.174)$$

and the constraint (5.166) implies that the scalars  $X_I$  are given by

$$X_I = u^2 q_I + \chi u^2 \left(-\frac{i}{\sqrt{2}\varrho^3} \left(\frac{1}{T}\frac{\partial\mathcal{G}}{\partial x^2} + \frac{\mathcal{H}}{T^2}\frac{\partial T}{\partial x^1}\right) + \frac{3\chi}{\varrho^4 u^2} V_J X^J\right) V_I \quad (5.175)$$

for constant  $q_I$ .

Lastly, consider the equations (4.34) and (4.35). These are equivalent to

$$\sqrt{2}\tilde{d} \left( c_4 \left(\frac{1}{f} \operatorname{Re} \lambda\right) - c_3 \left(\frac{1}{f} \operatorname{Re} \sigma\right) \right) = i_W d\Omega \quad (5.176)$$

and

$$\sqrt{2}\tilde{d} \left( c_3 \left(\frac{1}{f} \operatorname{Re} \lambda\right) + c_4 \left(\frac{1}{f} \operatorname{Re} \sigma\right) \right) = i_Y d\Omega + \sqrt{2}\varrho^2 \Omega. \quad (5.177)$$

where  $\tilde{d}$  denotes the restriction of the exterior derivative to hypersurfaces of constant  $t$ . The integrability conditions of these two equations are

$$\mathcal{L}_W d\Omega = 0 \quad (5.178)$$

and

$$\mathcal{L}_Y d\Omega = -\sqrt{2}\varrho^2 d\Omega \quad (5.179)$$

which hold automatically.

### 5.3 Solutions with $c_2 = c_3 = c_4 = 0$

We now turn to the class of solutions with  $c_2 = c_3 = c_4 = 0$ . It is clear that (4.19), (4.20) and (4.22) imply that  $K$  and  $Z$  are covariantly constant. In particular, this implies that  $K^2$  is constant and without loss of generality we can set  $K^2 = 1$ . One can choose co-ordinates  $\phi, \psi$  so that locally

$$K = \frac{\partial}{\partial\phi}, \quad Z = \frac{\partial}{\partial\psi}, \quad (5.180)$$

and two additional co-ordinates  $x^1, x^2$  can be chosen on the base so that the Kähler metric takes the form

$$ds^2 = d\phi^2 + d\psi^2 + T^2((dx^1)^2 + (dx^2)^2) \quad (5.181)$$

where  $T = T(x^1, x^2)$ .

Recall that  $K \wedge Z - \frac{1}{2}J$  is anti-self-dual, so taking positive orientation on the base with respect to  $T^2 d\phi \wedge d\psi \wedge dx^1 \wedge dx^2$ , we have

$$J = d\phi \wedge d\psi - T^2 dx^1 \wedge dx^2. \quad (5.182)$$

This complex structure is automatically covariantly constant. Then the real portions of (4.27) and (4.28) imply that

$$\mathcal{L}_K f = \mathcal{L}_Z f = 0, \quad (5.183)$$

so that  $f$  is only a function of  $x^1$  and  $x^2$ . Also, as in the previous case, the real part of (4.24) and (4.25) implies that

$$\mathcal{L}_K X^I = \mathcal{L}_Z X^I = 0, \quad (5.184)$$

so

$$X^I = X^I(x^1, x^2). \quad (5.185)$$

It is convenient to define

$$\mathcal{G} = \frac{2i}{f} \text{Im } \lambda, \quad \mathcal{H} = \frac{2i}{f} \text{Im } \sigma. \quad (5.186)$$

Then the remaining portions of (4.27), (4.28), (4.24) and (4.25) can be rewritten as

$$\begin{aligned} \frac{\partial}{\partial x^1} \left( \frac{X_I}{f^2} \right) &= \sqrt{2} i \chi T \mathcal{H} V_I, \\ \frac{\partial}{\partial x^2} \left( \frac{X_I}{f^2} \right) &= \sqrt{2} i \chi T \mathcal{G} V_I. \end{aligned} \quad (5.187)$$

As the  $V_I$  do not all vanish, these constraints imply that

$$\mathcal{G} = \mathcal{G}(x^1, x^2), \quad \mathcal{H} = \mathcal{H}(x^1, x^2). \quad (5.188)$$

We take the following holomorphic basis for the Kähler base space:

$$\begin{aligned} \mathbf{e}^1 &= \frac{1}{\sqrt{2}} (T dx^1 + i d\phi), \\ \mathbf{e}^2 &= \frac{1}{\sqrt{2}} (T dx^2 + i d\psi), \end{aligned} \quad (5.189)$$

with  $\mathbf{e}^{\bar{1}}, \mathbf{e}^{\bar{2}}$  fixed by complex conjugation.

In this basis, we obtain the following spin connection components:

$$\begin{aligned} \omega_{1,mn} \epsilon^{mn} &= \frac{1}{\sqrt{2} T^2} \frac{\partial T}{\partial x^2}, \\ \omega_{2,mn} \epsilon^{mn} &= -\frac{1}{\sqrt{2} T^2} \frac{\partial T}{\partial x^1} \end{aligned} \quad (5.190)$$

and the components of  $K$  and  $Z$  are:

$$\begin{aligned} K_1 &= -K_{\bar{1}} = -\frac{i}{\sqrt{2}}, & K_2 &= K_{\bar{2}} = 0, \\ Z_1 &= Z_{\bar{1}} = 0, & Z_2 &= -Z_{\bar{2}} = -\frac{i}{\sqrt{2}}. \end{aligned} \quad (5.191)$$

To proceed, we turn to the constraint (4.31). This implies that

$$d\Omega_{1\bar{1}} = d\Omega_{2\bar{2}} \quad (5.192)$$

and

$$d\Omega_{1\bar{2}} = -3\frac{\chi V_I X^I}{f^4}. \quad (5.193)$$

The constraints (4.36) and (4.37) then imply that

$$d\Omega_{1\bar{1}} = d\Omega_{2\bar{2}} = 0 \quad (5.194)$$

together with

$$d\Omega_{12} = -\frac{i}{\sqrt{2}}\left(\frac{1}{T}\frac{\partial\mathcal{G}}{\partial x^2} + \frac{\mathcal{H}}{T^2}\frac{\partial T}{\partial x^1}\right) \quad (5.195)$$

and the constraints

$$\frac{\partial}{\partial x^2}(T\mathcal{H}) = \frac{\partial}{\partial x^1}(T\mathcal{G}), \quad (5.196)$$

and

$$\frac{\partial}{\partial x^1}\left(\frac{\mathcal{H}}{T}\right) = \frac{\partial}{\partial x^2}\left(\frac{\mathcal{G}}{T}\right), \quad \frac{\partial}{\partial x^1}\left(\frac{\mathcal{G}}{T}\right) = -\frac{\partial}{\partial x^2}\left(\frac{\mathcal{H}}{T}\right). \quad (5.197)$$

Just as in the previous section, these constraints imply that  $T^{-1}\mathcal{H}$  and  $T^{-1}\mathcal{G}$  satisfy the Cauchy-Riemann equations;  $T^{-1}(\mathcal{H} + i\mathcal{G})$  is a holomorphic function of  $x^1 + ix^2$ .

In these co-ordinates  $d\Omega$  takes the form

$$d\Omega = -\frac{i}{\sqrt{2}}\left(T\frac{\partial\mathcal{G}}{\partial x^2} + \mathcal{H}\frac{\partial T}{\partial x^1}\right)\left(dx^1 \wedge dx^2 - \frac{1}{T^2}d\phi \wedge d\psi\right) - \frac{3\chi}{f^4}V_I X^I (T^2 dx^1 \wedge dx^2 + d\phi \wedge d\psi). \quad (5.198)$$

The gauge field strengths for these solutions can be written as

$$F^I = d(f^2 X^I(dt + \Omega)) + 6\chi\frac{T^2}{f^2}V_J\left(X^I X^J - \frac{1}{2}Q^{IJ}\right)dx^1 \wedge dx^2 \quad (5.199)$$

which automatically satisfy the Bianchi identity  $dF^I = 0$ .

Finally the integrability condition associated with (3.59) can be written as

$$\square \log T = 18\frac{\chi^2}{f^2}\left(X^I X^J - \frac{1}{2}Q^{IJ}\right)V_I V_J T^2. \quad (5.200)$$

It is then straightforward to show that these constraints imply that the integrability condition  $d^2\Omega = 0$  associated with the expression in (5.198) holds automatically. Lastly, the constraints (4.34) and (4.35) fix  $d(\text{Re}\frac{\lambda}{f})$  and  $d(\text{Re}\frac{\sigma}{f})$  in terms of constant linear combinations of  $d\phi$  and  $d\psi$ ; these conditions do not impose any further constraints on the geometry.

## 6. Summary and discussion

In this paper we have employed the spinorial geometry method for the task of classifying 1/2 supersymmetric solutions with at least one time-like Killing spinor of the theory of  $N = 2, D = 5$  supergravity. Our results provide a general framework for the explicit

construction of many new black holes and the investigation of their physical properties and relevance to AdS/CFT correspondence and holography.

In general, supersymmetric solutions in five dimensional theories must preserve either 2, 4, 6 or 8 of the supersymmetries. This is because the Killing spinor equations are linear over  $\mathbb{C}$ . However in the ungauged theories, it was found that supersymmetric solutions can only preserve 4 or 8 of supersymmetries [18, 31]. Moreover, to find time-like supersymmetric solutions in the ungauged theory, one must solve the gauge equations and the Bianchi identities in addition to the Killing spinor equations. In the null case one must additionally solve one of the components of the Einstein equations of motion. A similar situation arises for 1/4 supersymmetric solutions in the gauged theory. However, for the solutions with 1/2 supersymmetry considered in our present work, we have demonstrated that if one of the Killing spinors is time-like, then supersymmetry and Bianchi identities alone imply that all components of Einstein and gauge equations together with the scalar equations are automatically satisfied.

Maximally supersymmetric solutions (preserving all 8 of the supersymmetries) of the five dimensional gauged supergravity theory have vanishing gauge field strengths and constant scalars and are locally isometric to  $AdS_5$ . Moreover, it has been demonstrated in [32] that all solutions of  $N = 2$ ,  $D = 5$  supergravity preserving 3/4 of supersymmetry must be locally isometric to  $AdS_5$ , with vanishing gauge field strengths and constant scalars. An analogous situation also arises in the case of  $D = 11$  supergravity, where it has been shown that all solutions with 31/32 supersymmetry must be locally isometric to a maximally supersymmetric solution [33]. However, in the case of  $D = 11$  supergravity, it has been shown that one cannot obtain 31/32 supersymmetric solutions by taking quotients of the maximally supersymmetric solutions [34]. In contrast, there is a 3/4-supersymmetric supersymmetric solution of  $N = 2$ ,  $D = 5$  gauged supergravity which is obtained by taking a certain quotient of  $AdS_5$  [35].

One future direction is the completion of the classification of supersymmetric solutions in  $N = 2$ ,  $D = 5$  supergravity by classifying 1/2 supersymmetric solutions with two null Killing spinors (i.e., two Killing spinors with associated null Killing vectors). In addition, it would be interesting to investigate whether there are any regular asymptotically  $AdS_5$  black ring solutions (see [36] for a recent discussion). Supersymmetric black rings are known to exist in the ungauged theory [37–40]. For asymptotically flat supersymmetric black rings of  $N = 2$ ,  $D = 5$  ungauged supergravity, supersymmetry is enhanced from 1/2 supersymmetry to maximal supersymmetry at the horizon. If there do exist  $AdS_5$  black rings in the gauged theory, it may be reasonable to expect that the supersymmetry will be enhanced from 1/4 to 1/2 at the horizon. We hope that the classification of 1/2 supersymmetric solutions can provide a method of determining whether there exists a 1/2 supersymmetric solution corresponding to the near-horizon geometry of a black ring. It should be noted that in [36], black rings with two  $U(1)$  symmetries have already been excluded. However, as we have seen, 1/2 supersymmetric solutions in general only have one additional  $U(1)$  symmetry; so more general ring solutions may be possible.

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## A. Systematic treatment of the Killing spinor equation

In this appendix we will evaluate the linear system obtained from the Killing spinor acting on  $\eta^a$  given in (3.23), keeping the parameters arbitrary. It would naively appear that we have to evaluate two sets of Killing spinor equations, according to the two choices of symplectic index  $a$ . However, making use of the symplectic Majorana condition, together with the fact that the gauge field strengths and scalars are real, it is straightforward to show that it suffices just to consider the case when  $a = 1$ , the  $a = 2$  equations are then implied automatically. In the following, it will be convenient to define  $H = X_I F^I$  and  $\mu_{\bar{p}} = \delta_{\bar{p}q} \mu^q$ .

From the dilatino equation we obtain

$$\begin{aligned} & \sigma (F^I{}_{mn} - X^I H_{mn}) \epsilon^{mn} - \lambda (F^I{}_m{}^m - X^I H_m{}^m + \partial_0 X^I) = \\ & 2\chi V_J \left( X^I X^J - \frac{3}{2} Q^{IJ} \right) \sigma^* + \sqrt{2}i (F^I{}_{0m} - X^I H_{0m} - \partial_m X^I) \mu^m \end{aligned} \quad (\text{A.1})$$

$$\begin{aligned} & \sqrt{2}i\sigma (F^I{}_{0\bar{m}} \epsilon^{\bar{m}\bar{q}} - X^I H_{0\bar{m}} + \partial_{\bar{m}} X^I) \epsilon^{\bar{m}\bar{q}} - \mu_{\bar{q}} (F^I{}_m{}^m - X^I H_m{}^m - \partial_0 X^I) = \\ & 2\chi V_J \left( X^I X^J - \frac{3}{2} Q^{IJ} \right) \epsilon_{\bar{m}\bar{q}} (\mu^m)^* - \sqrt{2}i\lambda (F^I{}_{0\bar{q}} - X^I H_{0\bar{q}} + \partial_{\bar{q}} X^I) \\ & \qquad \qquad \qquad - 2 (F^I{}_{m\bar{q}} - X^I H_{m\bar{q}}) \mu^m \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} & \sqrt{2}i (-F^I{}_{0\bar{m}} + X^I H_{0\bar{m}} + \partial_{\bar{m}} X^I) \epsilon^{\bar{m}n} \mu^n - \lambda \epsilon^{\bar{m}\bar{n}} (F^I{}_{\bar{m}\bar{n}} - X^I H_{\bar{m}\bar{n}}) = \\ & -2\chi V_J \left( X^I X^J - \frac{3}{2} Q^{IJ} \right) \lambda^* - \sigma (F^I{}_m{}^m - \partial_0 X^I - X^I H_m{}^m) \end{aligned} \quad (\text{A.3})$$

Then from the gravitino part of the Killing spinor equations we obtain the following constraints:

From along the 0-direction of the supercovariant derivative-

$$\begin{aligned} \partial_0 \lambda &= -\frac{i\mu^m}{\sqrt{2}} (-\omega_{0,0m} + H_{0m}) - \frac{1}{4} \sigma (2\omega_{0,mn} + H_{mn}) \epsilon^{mn} \\ &+ \frac{1}{4} \lambda (H_m{}^m + 2\omega_{0,m}{}^m) - \frac{\chi}{2} V_I (X^I - 3A_0^I) \sigma^*, \end{aligned} \quad (\text{A.4})$$

$$\begin{aligned} \partial_0 \mu_{\bar{q}} &= -\frac{i\sigma}{\sqrt{2}} (\omega_{0,0\bar{m}} \epsilon^{\bar{m}\bar{q}} + H_{0\bar{m}} \epsilon^{\bar{m}\bar{q}}) - \frac{i}{\sqrt{2}} \lambda (\omega_{0,0\bar{q}} + H_{0\bar{q}}) - \frac{1}{4} (H_m{}^m - 2\omega_{0,m}{}^m) \mu_{\bar{q}} \\ &- \left( -\frac{1}{2} H_{m\bar{q}} + \omega_{0,m\bar{q}} \right) \mu^m + \frac{\chi}{2} V_I (X^I + 3A_0^I) \epsilon_{\bar{m}\bar{q}} (\mu^m)^*, \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned} \partial_0 \sigma &= -\frac{i}{\sqrt{2}} (-\omega_{0,0\bar{m}} + H_{0\bar{m}}) \epsilon^{\bar{m}n} \mu^n - \frac{1}{4} \sigma (2\omega_{0,m}{}^m + H_m{}^m) \\ &+ \frac{1}{4} \lambda (H_{\bar{m}\bar{n}} + 2\omega_{0,\bar{m}\bar{n}}) \epsilon^{\bar{m}\bar{n}} + \frac{\chi}{2} V_I (X^I - 3A_0^I) \lambda^*. \end{aligned} \quad (\text{A.6})$$



From along the  $p$ -direction of the supercovariant derivative-

$$\partial_p \lambda = \frac{i}{\sqrt{2}} \left( \omega_{p,0m} - \frac{3}{2} H_{pm} \right) \mu^m - \frac{\sigma}{2} \omega_{p,mn} \epsilon^{mn} + \frac{\lambda}{2} \left( \omega_{p,m}{}^m - \frac{3}{2} H_{0p} \right) + \frac{3\chi}{2} V_I A_p^I \sigma^*, \quad (\text{A.7})$$

$$\begin{aligned} \partial_p \mu_{\bar{q}} &= \frac{i\sigma}{2\sqrt{2}} \left( -2\omega_{p,0\bar{m}} \epsilon^{\bar{m}\bar{q}} + \frac{1}{2} H_{m\bar{n}} \epsilon^{m\bar{n}} \delta_{p\bar{q}} \right) - \frac{i\lambda}{2\sqrt{2}} (2\omega_{p,0\bar{q}} + 3H_{p\bar{q}} - H_m{}^m \delta_{p\bar{q}}) \\ &+ \left( \frac{1}{2} \omega_{p,m}{}^m + \frac{3}{4} H_{0p} \right) \mu_{\bar{q}} - \mu^m \left( \omega_{p,m\bar{q}} + \frac{1}{2} H_{0m} \delta_{p\bar{q}} \right) \\ &+ \chi V_I \left( -\frac{i}{\sqrt{2}} X^I \sigma^* \delta_{p\bar{q}} + \frac{3}{2} A_p^I \epsilon_{\bar{m}\bar{q}} (\mu^m)^* \right), \end{aligned} \quad (\text{A.8})$$

$$\begin{aligned} \partial_p \sigma &= \frac{i}{2\sqrt{2}} (2\omega_{p,0\bar{m}} - H_{p\bar{m}}) \epsilon^{\bar{m}n} \mu^n + \frac{\lambda}{2} (\omega_{p,\bar{m}\bar{n}} \epsilon^{\bar{m}\bar{n}} - H_{0\bar{n}} \epsilon^{\bar{n}p}) - \frac{\sigma}{4} (2\omega_{p,m}{}^m + H_{0p}) \\ &- \frac{i}{2\sqrt{2}} H_m{}^m \epsilon_{p\bar{n}} \mu_{\bar{n}} - \chi V_I \left( \frac{i}{\sqrt{2}} X^I (\mu_{\bar{p}})^* + \frac{3}{2} A_p^I \lambda^* \right). \end{aligned} \quad (\text{A.9})$$

From along the  $\bar{p}$ -direction of the supercovariant derivative-

$$\begin{aligned} \partial_{\bar{p}} \lambda &= \frac{i}{2\sqrt{2}} (2\omega_{\bar{p},0m} - H_{\bar{p}m}) \mu^m + \frac{\lambda}{4} (2\omega_{\bar{p},m}{}^m - H_{0\bar{p}}) + \frac{1}{2} \sigma (H_{0m} \epsilon^m{}_{\bar{p}} - \omega_{\bar{p},mn} \epsilon^{mn}) \\ &+ \frac{i}{2\sqrt{2}} H_m{}^m \mu_{\bar{p}} + \chi V_I \left( \frac{i}{\sqrt{2}} X^I \epsilon_{\bar{p}\bar{m}} (\mu^m)^* + \frac{3}{2} A_{\bar{p}}^I \sigma^* \right), \end{aligned} \quad (\text{A.10})$$

$$\begin{aligned} \partial_{\bar{p}} \mu_{\bar{q}} &= -\frac{i}{\sqrt{2}} \lambda \left( \omega_{\bar{p},0\bar{q}} + \frac{1}{4} H_{\bar{m}\bar{n}} \epsilon^{\bar{m}\bar{n}} \epsilon_{\bar{p}\bar{q}} \right) - \frac{i}{\sqrt{2}} \sigma \left( \omega_{\bar{p},0m} \epsilon^m{}_{\bar{q}} + \frac{1}{2} H_m{}^m \epsilon_{\bar{p}\bar{q}} + \frac{3}{2} H_{\bar{p}m} \epsilon^m{}_{\bar{q}} \right) \\ &- \mu^n \left( \omega_{\bar{p},n\bar{q}} + \frac{1}{2} H_{0\bar{m}} \epsilon^{\bar{m}n} \epsilon_{\bar{p}\bar{q}} \right) + \left( \frac{3}{4} H_{0\bar{p}} + \frac{1}{2} \omega_{\bar{p},m}{}^m \right) \mu_{\bar{q}} \\ &+ \chi V_I \left( \frac{i}{\sqrt{2}} X^I \lambda^* \epsilon_{\bar{p}\bar{q}} + \frac{3}{2} A_{\bar{p}}^I \epsilon_{\bar{m}\bar{q}} (\mu^m)^* \right), \end{aligned} \quad (\text{A.11})$$

$$\begin{aligned} \partial_{\bar{p}} \sigma &= \frac{i}{\sqrt{2}} \omega_{\bar{p},0\bar{m}} \epsilon^{\bar{m}n} \mu^n + \frac{1}{2} \lambda \omega_{\bar{p},\bar{m}\bar{n}} \epsilon^{\bar{m}\bar{n}} - \sigma \left( \frac{1}{2} \omega_{\bar{p},m}{}^m + \frac{3}{4} H_{0\bar{p}} \right) \\ &+ \frac{3i}{4\sqrt{2}} H_{\bar{m}\bar{n}} \epsilon^{\bar{m}\bar{n}} \mu_{\bar{p}} - \frac{3\chi}{2} V_I A_{\bar{p}}^I \lambda^*. \end{aligned} \quad (\text{A.12})$$

Throughout these equations, spatial indices of  $\epsilon_{mn}$ ,  $\epsilon_{\bar{m}\bar{n}}$ ,  $\omega_{A,BC}$ ,  $F^I{}_{AB}$  and  $H_{AB}$  have been raised with  $\delta^{p\bar{q}}$ .

## B. Integrability conditions and equations of motion

In this appendix we examine the integrability conditions of the Killing spinor equations. It will be shown that, if a background preserves at least half of the supersymmetry, and admits a Killing spinor for which the associated Killing vector is time-like, and the Bianchi identity holds, then all components of the Einstein, gauge and scalar equations hold automatically.

First we consider the integrability condition associated with the gravitino equation

(2.8). After some gamma matrix manipulation we find

$$\begin{aligned}
0 = & -\frac{1}{4}R_{\alpha\beta\beta_1\beta_2}\gamma^{\beta_1\beta_2}\epsilon^a - \frac{1}{4}\nabla_{[\alpha}X_I(\gamma_{\beta]}^{\beta_1\beta_2} - 4\delta_{[\alpha}^{\beta_1}\gamma^{\beta_2]})F^I{}_{\beta_1\beta_2}\epsilon^a \\
& + \frac{1}{4}X_I(\gamma_{[\alpha}^{\beta_1\beta_2} - 4\delta_{[\alpha}^{\beta_1}\gamma^{\beta_2]})\nabla_{\beta]}F^I{}_{\beta_1\beta_2}\epsilon^a + \chi V_I\left(\nabla_{[\alpha}X^I\gamma_{\beta]} - \frac{3}{2}F^I{}_{\alpha\beta}\right)\epsilon^{ab}\epsilon^b \\
& + \frac{1}{4}X_I X_J\left(F^I{}_{\beta_1\beta_2}F^J{}_{\beta_3[\alpha}\gamma_{\beta]}^{\beta_1\beta_2\beta_3} + F^I{}_{\beta_1[\alpha}F^J\beta_1\beta_2\gamma_{\beta]\beta_2}\right. \\
& \left. + \frac{1}{4}F^I{}_{\beta_1\beta_2}F^J\beta_1\beta_2\gamma_{\alpha\beta} - \frac{3}{2}F^I{}_{\alpha\beta_1}F^J{}_{\beta_2}\gamma^{\beta_1\beta_2}\right)\epsilon^a \\
& + \frac{\chi}{4}V_I X^I X_J(F^J{}_{\beta_1\beta_2}\gamma_{\alpha\beta}{}^{\beta_1\beta_2} + 4\gamma^\mu{}_{[\alpha}F^J{}_{\beta]\mu})\epsilon^{ab}\epsilon^b + \frac{\chi^2}{2}V_I V_J X^I X^J \gamma_{\alpha\beta}\epsilon^a. \quad (\text{B.1})
\end{aligned}$$

Next consider the dilatino equation (2.13). This gives the integrability condition

$$\begin{aligned}
0 = & \frac{3}{4}\gamma^\beta\nabla_\alpha\nabla_\beta X_I\epsilon^a + \frac{3\chi}{2}\left(\nabla_\alpha(X_I V_J X^J) + \frac{1}{2}V_J X^J\nabla_\beta X_I\gamma^\beta{}_\alpha\right)\epsilon^{ab}\epsilon^b \\
& + \nabla_\alpha\left(\left(\frac{1}{4}Q_{IJ} - \frac{3}{8}X_I X_J\right)F^J{}_{\beta_1\beta_2}\right)\gamma^{\beta_1\beta_2}\epsilon^a \\
& + \left(\frac{3}{16}X_J F^J{}_{\beta_1\beta_2}\nabla_{\beta_3}X_I\gamma_\alpha{}^{\beta_1\beta_2\beta_3} - \frac{3}{4}X_J F^J{}_{\alpha\beta_1}\nabla_{\beta_2}X_I\gamma^{\beta_1\beta_2}{}_\alpha\right)\epsilon^a \\
& - 2\chi V_K X^K\left(\frac{1}{4}Q_{IJ} - \frac{3}{8}X_I X_J\right)F^J{}_{\alpha\beta}\gamma^\beta\epsilon^{ab}\epsilon^b \\
& + X_K\left(\left(\frac{1}{8}Q_{IJ} - \frac{3}{16}X_I X_J\right)F^J{}_{\alpha\beta_1}F^K{}_{\beta_2\beta_3}\gamma^{\beta_1\beta_2\beta_3}{}_\alpha\right. \\
& \left. - \frac{1}{16}C_{IJM}X^M F^J{}_{\beta_1\mu}F^K{}_{\beta_2}{}^\mu\gamma_\alpha{}^{\beta_1\beta_2} + \left(\frac{1}{2}Q_{IJ} - \frac{3}{4}X_I X_J\right)F^J{}_{\beta\mu}F^K{}_\alpha{}^\mu\gamma^\beta\right)\epsilon^a. \quad (\text{B.2})
\end{aligned}$$

It will be convenient to define

$$\begin{aligned}
E_{\alpha\beta} = & R_{\alpha\beta} + Q_{IJ}F^I{}_{\alpha\mu}F^J{}_{\beta}{}^\mu - Q_{IJ}\nabla_\alpha X^I\nabla_\beta X^J \\
& + g_{\alpha\beta}\left(-\frac{1}{6}Q_{IJ}F^I{}_{\beta_1\beta_2}F^J\beta_1\beta_2 + 6\chi^2\left(\frac{1}{2}Q^{IJ} - X^I X^J\right)V_I V_J\right) \\
G_{I\alpha} = & \nabla^\beta(Q_{IJ}F^J{}_{\alpha\beta}) + \frac{1}{16}C_{IJK}\epsilon_\alpha{}^{\beta_1\beta_2\beta_3\beta_4}F^J{}_{\beta_1\beta_2}F^K{}_{\beta_3\beta_4} \\
S_I = & \nabla^\alpha\nabla_\alpha X_I - \left(\frac{1}{6}C_{MNI} - \frac{1}{2}X_I C_{MNJ}X^J\right)\nabla_\alpha X^M\nabla^\alpha X^N \\
& - \frac{1}{2}\left(X_M X^P C_{NPI} - \frac{1}{6}C_{MNI} - 6X_I X_M X_N + \frac{1}{6}X_I C_{MNJ}X^J\right)F^M{}_{\beta_1\beta_2}F^N\beta_1\beta_2 \\
& - 3\chi^2 V_M V_N\left(\frac{1}{2}Q^{ML}Q^{NP}C_{LPI} + X_I(Q^{MN} - 2X^M X^N)\right) \quad (\text{B.3})
\end{aligned}$$

so that  $E_{\alpha\beta} = 0$ ,  $G_{I\alpha} = 0$  and  $S_I = 0$  correspond to the Einstein, gauge field and scalar equations of motion respectively.

To proceed we act on the gravitino integrability condition (B.1) from the left with  $\gamma^\beta$  and contract over the index  $\beta$ . We assume that the Bianchi identity  $dF^I = 0$  holds. After

some considerable gamma matrix manipulation (and making use of (2.13) to simplify the expressions further), we find the constraint

$$\left(E_{\alpha\beta}\gamma^\beta + \frac{1}{3}X^I\left(\gamma_\alpha{}^\beta G_{I\beta} - 2G_{I\alpha}\right)\right)\epsilon^a = 0. \quad (\text{B.4})$$

Also, on contracting the dilatino integrability condition (B.2) with  $\gamma^\alpha$ , and again assuming the Bianchi identity  $dF^I = 0$  holds, we find

$$\left(S_I - \frac{2}{3}(G_{I\alpha} - X_I X^J G_{J\alpha})\gamma^\alpha\right)\epsilon^a = 0. \quad (\text{B.5})$$

To proceed, we evaluate the constraints (B.4) and (B.5) on a background which preserves at least half of the supersymmetry, and which admits a Killing spinor for which the associated Killing vector is time-like. In particular, we first consider a generic Killing spinor

$$\eta^1 = \lambda 1 + \mu^i e^i + \sigma e^{12}, \quad (\text{B.6})$$

$$\eta^2 = -\sigma^* 1 - \epsilon_{ij}(\mu^i)^* e^j + \lambda^* e^{12}. \quad (\text{B.7})$$

Substituting this expression into (B.4) for  $\alpha = 0$  gives the constraints

$$\begin{aligned} \lambda E_{00} - \sqrt{2}i\mu^p \left(E_{0p} + \frac{1}{3}X^I G_{Ip}\right) - \frac{2}{3}\lambda X^I G_{I0} &= 0, \\ E_{00}\mu^p + \sqrt{2}i\sigma \left(E_{0q} - \frac{1}{3}X^I G_{Iq}\right)\epsilon^{qp} + \sqrt{2}i\lambda \left(E_{0p} - \frac{1}{3}X^I G_{Ip}\right) + \frac{2}{3}X^I G_{I0}\mu^p &= 0, \\ \sigma E_{00} - \sqrt{2}i \left(E_{0\bar{p}} + \frac{1}{3}X^I G_{I\bar{p}}\right)\epsilon^{\bar{p}q}\mu^q - \frac{2}{3}X^I G_{I0}\sigma &= 0. \end{aligned} \quad (\text{B.8})$$

Evaluating these constraints on the canonical form of the  $N = 1$  time-like Killing spinor by setting  $\lambda = f, \mu^1 = \mu^2 = \sigma = 0$  we obtain the constraints

$$\begin{aligned} E_{00} &= \frac{2}{3}X^I G_{I0}, \\ E_{0p} &= \frac{1}{3}X^I G_{Ip}, \\ E_{0\bar{p}} &= \frac{1}{3}X^I G_{I\bar{p}}. \end{aligned} \quad (\text{B.9})$$

Now substitute these expressions back into (B.8) and eliminate the  $E_{\alpha\beta}$  terms to find

$$\begin{aligned} X^I G_{Ip}\mu^p &= 0, \\ X^I G_{I0}\mu^p &= 0, \\ X^I G_{I\bar{p}}\epsilon^{\bar{p}q}\mu^q &= 0. \end{aligned} \quad (\text{B.10})$$

Assuming that the background is at least half-supersymmetric, we take  $(\mu^1, \mu^2) \neq (0, 0)$ , and hence find

$$X^I G_{I\alpha} = 0 \quad (\text{B.11})$$

and

$$E_{00} = E_{0p} = 0. \quad (\text{B.12})$$

Next consider (B.4) for  $\alpha = p$  evaluated on the generic spinor  $\eta^1$ . Using the constraints  $E_{0p} = 0$  and  $X^I G_{I\alpha} = 0$  which we have already obtained, this expression simplifies to

$$(E_{pq}\gamma^q + E_{p\bar{q}}\gamma^{\bar{q}})\eta^1 = 0 \quad (\text{B.13})$$

from which we find the constraints

$$\begin{aligned} E_{pq}\mu^q &= 0, \\ \sigma E_{pq}\epsilon^q_{\bar{\ell}} + \lambda E_{p\bar{\ell}} &= 0, \\ E_{p\bar{q}}\epsilon^{\bar{q}}_{\ell}\mu^{\ell} &= 0 \end{aligned} \quad (\text{B.14})$$

and taking (B.4) with  $\alpha = p$  we find

$$\begin{aligned} E_{\bar{p}q}\mu^q &= 0, \\ \sigma E_{\bar{p}q}\epsilon^q_{\bar{\ell}} + \lambda E_{\bar{p}\bar{\ell}} &= 0, \\ E_{\bar{p}\bar{q}}\epsilon^{\bar{q}}_{\ell}\mu^{\ell} &= 0. \end{aligned} \quad (\text{B.15})$$

Evaluating these constraints on the canonical  $N = 1$  time-like spinor by taking  $\lambda = f, \mu^1 = \mu^2 = \sigma = 0$ , we obtain the constraints

$$E_{p\bar{q}} = 0, \quad E_{pq} = 0. \quad (\text{B.16})$$

Hence we have shown that for solutions with at least half supersymmetry, the constraint (B.4) implies that

$$E_{\alpha\beta} = 0, \quad X^I G_{I\alpha} = 0. \quad (\text{B.17})$$

Next consider the constraint (B.5) obtained from the dilatino integrability conditions. On using  $X^I G_{I\alpha} = 0$  this constraint simplifies to

$$\left(S_I - \frac{2}{3}G_{I\alpha}\gamma^\alpha\right)\epsilon^a = 0. \quad (\text{B.18})$$

Evaluating this expression on the generic Killing spinor  $\eta^a$ , one obtains

$$\begin{aligned} \lambda \left(S_I - \frac{2}{3}G_{I0}\right) + \frac{2\sqrt{2}i}{3}G_{Ip}\mu^p &= 0, \\ S_I\mu^p + \frac{2}{3}G_{I0}\mu^p + \frac{2\sqrt{2}i}{3}(\sigma G_{Iq}\epsilon^{qp} + \lambda G_I^p) &= 0, \\ \sigma \left(S_I - \frac{2}{3}G_{I0}\right) + \frac{2\sqrt{2}i}{3}G_{I\bar{p}}\epsilon^{\bar{p}}_q\mu^q &= 0. \end{aligned} \quad (\text{B.19})$$

Evaluating these constraints on the canonical  $N = 1$  time-like spinor by taking  $\lambda = f, \mu^1 = \mu^2 = \sigma = 0$ , the following conditions are obtained

$$\begin{aligned} S_I &= \frac{2}{3}G_{I0}, \\ G_{Ip} &= G_{I\bar{p}} = 0. \end{aligned} \quad (\text{B.20})$$

Now substitute these constraints back into (B.19) to find

$$G_{I0}\mu^p = 0. \tag{B.21}$$

Assuming that the background is at least half-supersymmetric, we take  $(\mu^1, \mu^2) \neq (0, 0)$ , and hence

$$G_{I0} = 0. \tag{B.22}$$

Hence from the constraint (B.5) we have found the constraints

$$S_I = 0, \quad G_{I\alpha} = 0. \tag{B.23}$$

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